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On some basic applications of Gröbner Bases

Streszczenie. In this paper, we introduced notions and basic facts on Gröbner Basis Theory. An application of the theory for solving algebraic equations is a fundamental result presented in the paper. Methods of elimination of parameter from a parametric equation for determining a polynomial implicit form of an equation is the second one result presented in the paper, the methods are illustrated with curves: the Folium of Descartes, Bézier's curve and many surfaces, especially the Enneper surface.

1. Notation and basic facts

We introduce basic notions which will be used throughout this paper. Let $\Bbbk[x_1,\ldots,x_n]$ be a polynomial ring over an infinite field $\Bbbk$. Denote

$$X^\alpha := X_1^{\alpha_1}\cdots X_n^{\alpha_n}.$$

For $f=\sum\limits_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha \in \Bbbk[x_1,\ldots,x_n]$ the *support* of f is

$$\operatorname{supp} f = \{\alpha\,:\,c_\alpha\neq 0\}.$$

and for $P\subset \Bbbk[x_1,\ldots,x_n]$ we put $\operatorname{supp} P = \bigcup\limits_{f\in P}\operatorname{supp} f$. Let T be a set of all monomials in $\Bbbk[x_1,\ldots,x_n]$. *An admissible term ordering* $\prec$ is a linear ordering defined on the set T that satisfies the following conditions:

(i) $1\prec t$ for all $t\in T$,

(ii) $t_1\prec t_2$ implies $t_1\cdot s\prec t_2\cdot s$ for all $s,t_1,t_2\in T$.

Under the natural correspondence between terms and exponent tuples, we define admissible term ordering in $\mathbb{N}^n$ in the following way:

$$\alpha\prec\beta \text{ if } X^\alpha\prec X^\beta.$$

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Let $\prec$ be a fixed admissible term ordering in $\mathbb{N}^n$. For a nonzero $f \in \Bbbk[x_1, \dots, x_n]$ we define the *exponent*, the *leading monomial*, the *leading coefficient*, the *leading term* of f as:

$$\begin{aligned} \exp_\prec f :=& \max_\prec\{\alpha \,:\, \alpha \in \operatorname{supp} f\}, \\ \mathrm{LM}_\prec f :=& X^{\exp_\prec f}, \\ \mathrm{LC}_\prec f :=& c_{\exp_\prec f}, \\ \mathrm{LT}_\prec f :=& (\,\mathrm{LC}_\prec f)(\mathrm{LM}_\prec f), \end{aligned}$$

respectively, and for any $P \subset \Bbbk[x_1, \dots, x_n] \setminus \{0\}$ we define the *leading monoid* and *standard monomials*:

$$\begin{aligned} \mathcal{L}_\prec(P) :=& \begin{cases} \bigcup\limits_{f \in P} (\exp_\prec f + \mathbb{N}^n) & \text{if } P \nsubseteq \{0\} \\ \varnothing & \text{if } P \subseteq \{0\} \end{cases} \\ \mathcal{D}_P :=& \mathbb{N}^n \setminus \mathcal{L}_\prec(P). \end{aligned}$$

Let $P \subset \Bbbk[x_1, \dots, x_n] \setminus \{0\}$. The (polynomial) *ideal generated by* P we denote by $\langle P \rangle$.

The fundamental concept of the theory is the definition of a finite set uniquely defining a polynomial ideal. Let starts with preliminary definitions.

Definition 1

A finite subset $G \subset \Bbbk[x_1, \dots, x_n]$ is a *Gröbner basis* with respect to fixed admissible order $\prec$ if $0 \notin G$ and

$$\mathcal{L}_\prec(\langle G \rangle) = \mathcal{L}_\prec(G).$$

Definition 2

A *reduced Gröbner basis* of a polynomial ideal I is a Gröbner basis G of I such that:

(i) $\mathrm{LC}(p) = 1$ for all $p \in G$,

(ii) For all $p \in G$, no monomial of p lies in $\langle \mathrm{LT}(G \setminus \{p\}) \rangle$.

Definition 3

Let $f,\ g \in \Bbbk[x_1, \dots, x_n] \setminus \{0\}$.

(i) If $\mathrm{LM}(f) = x^\alpha$ and $\mathrm{LM}(g) = x^\beta$, then let $\gamma = \gamma_1 \cdots \gamma_n$, where $\gamma_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, \dots, n$. We call x^γ the least common multiple of $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$ and denote $x^\gamma = \mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g))$.

(ii) The $S-$polynomial of f and g is the combination

$$S(f,g) := \frac{x^\gamma}{\mathrm{LT}(f)} f - \frac{x^\gamma}{\mathrm{LT}(g)} g.$$

The following property of a polynomial ideal is a basic characterization of Gröbner basis, for a fixed monomial order.

Theorem 4 (Division Algorithm)
Fix an admissible term ordering in $\mathbb{N}^n$ and let $G = (g_1, \dots, g_s)$ be an ordered s–tuple of polynomials in $\Bbbk[x_1, \dots, x_n]$. Then every $f \in \Bbbk[x_1, \dots, x_n]$ can be written as

$$f = a_1 g_1 + \cdots + a_s g_s + r,$$

where $a_i, r \in \Bbbk[x_1 \dots, x_n]$, and either $r = 0$ or r is a linear combination, with coefficients in $\Bbbk$, of monomials, none of which is divisible by any of $LM(f_1), \dots, LM(f_s)$. We will call r a remainder of f on division by G.

Theorem 5 (Buchberger's Criterion)
Let I be a polynomial ideal. Then a basis $G = \{g_1, \dots, g_s\}$ of I (i.e. $I = \langle G \rangle$) is a Gröbner basis of I iff for all pairs $i \neq j$, the remainder of division of $S(g_i, g_j)$ by G (listed in some monomial order) is zero.

One of the fundamental theorems in computational commutative algebra is Buchberger's algorithm. The algorithm is an effective method for computation a Gröbner basis for a given set of generators of a polynomial ideal. Buchberger's algorithm, which is based on Dixon's lemma, is a generalization of the Euclidean algorithm and the Gaussian elimination.

Theorem 6 (Buchberger's Algorithm)
Let $f_1, \dots, f_s \in \Bbbk[x_1, \dots, x_n]$. Then there exists an algorithm for a computation of a Gröbner basis G of the ideal $\langle f_1, \dots, f_s \rangle$ such that

$$\langle f_1, \dots, f_s \rangle = \langle G \rangle.$$

We are interested in an effective algorithmic elimination of variables in polynomial equations. The classical elimination theory, which starts with Bézout's, is based on the theory of determinants, especially the *resultant*. We present the basic theorem of the modern theory of elimination in the computational commutative algebra. Let us start with the definition:

Definition 7
For $I = \langle f_1, \dots, f_s \rangle \subset \Bbbk[x_1, \dots, x_n]$ and a fixed $0 \leq l \leq n$, the $l-th$ *elimination ideal* is the ideal I_l in $\Bbbk[x_1, \dots, x_n]$ defined by

$$I_l := I \cap \Bbbk[x_{l+1}, \dots, x_n].$$

Theorem 8 (The Elimination Theorem)
Let $I \subset \Bbbk[x_1, \dots, x_n]$ be an ideal and let set G be a Gröbner basis of I with respect to the lex order. Then, for every $0 \leq l \leq n$, the set

$$G_l = G \cap \Bbbk[x_{l+1}, \dots, x_n]$$

is a Gröbner basis of the $l-th$ elimination ideal I_l.

A geometric interpretation, in complex affine $n-$dimensional space, of the elimination of one variable in a polynomial equation of complex variable gives the following theorem:

Corollary 9 ([1], Corollary 4, p. 127)
Let $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$ and assume that for some i the polynomial f_i is of the form

$$f_i = cx_1^N + \textit{terms in which } \deg x_1 < N,$$

where $c \in \mathbb{C}$ is nonzero and $N > 0$. If I_1 is the first elimination ideal, then in $\mathbb{C}^{n-1}$

$$\pi_1(V) = \boldsymbol{V}(I_1),$$

where π_1 is the canonical projection on the last $n-1$ components.

Let us recall that $V(I)$ is the set of common zeros of polynomials from I. An interpretation, in the affine space $\Bbbk^n$, of the elimination algorithm gives the following theorem:

Theorem 10 ([1], Theorem 1, p. 130)
Let $\Bbbk$ be an infinite field and let $F : \Bbbk^m \to \Bbbk^n$ be a polynomial mapping

$$\begin{aligned} x_1 &= f_1(t_1, \ldots, t_m) \\ &\vdots \\ x_n &= f_n(t_1, \ldots, t_m). \end{aligned} \tag{1}$$

Let $I = \langle x_1 - f_1, \ldots, x_n - f_n \rangle \subset \Bbbk[t_1, \ldots, t_m, x_1, \ldots, x_n]$ and $I_m = I \cap \Bbbk[x_1, \ldots, x_n]$ be the $m-$th elimination ideal. Then $\boldsymbol{V}(I_m)$ is the smallest variety in $\Bbbk^n$ containing $F(\Bbbk^m)$.

The previous results may be generalized to the case of rational mappings.

Definition 11
For $f_1, \ldots, f_n, g_1, \ldots, g_n \in \Bbbk[t_1, \ldots, t_m]$ a *rational parametrization* is

$$\begin{aligned} x_1 &= \tfrac{f_1(t_1, \ldots, t_m)}{g_1(t_1, \ldots, t_m)} \\ &\vdots \\ x_n &= \tfrac{f_n(t_1, \ldots, t_m)}{g_n(t_1, \ldots, t_m)}. \end{aligned}$$

For $W = \mathbf{V}(g_1 \cdots g_n) \subset \Bbbk^m$ we define the mapping

$$F : \Bbbk^m \setminus W \to \Bbbk^n$$

as follows:

$$F(t_1, \ldots, t_m) = \left(\frac{f_1(t_1, \ldots, t_m)}{g_1(t_1, \ldots, t_m)}, \ldots, \frac{f_n(t_1, \ldots, t_m)}{g_n(t_1, \ldots, t_m)} \right).$$

For

$$i(t_1, \ldots, t_m) := (t_1, \ldots, t_m, f_1(t_1, t_2, \ldots, t_m), \ldots, f_n(t_1, \ldots, t_m))$$

and

$$\pi_m(t_1, \ldots, t_m, x_1, \ldots, x_n) := (x_1, \ldots, x_n)$$

we have the relation

$$\begin{array}{ccc} & \mathbb{k}^{n+m} & \\ {\scriptstyle i} \nearrow & & \searrow {\scriptstyle \pi_m} \\ \mathbb{k}^m \setminus W & \xrightarrow[F]{} & \mathbb{k}^n \end{array}$$

Since $F = \pi_m \circ i$, so $F(\mathbb{k}^m) = \pi_m(V)$, where $V = i(\mathbb{k}^m)$.

Lemma 12
Let

$$x_i = \frac{f_i(t)}{g_i(t)}, \ \ i = 1, \ldots, n$$

be a rational parametrization of one parameter t such that polynomials $g_i(t)$ and $f_i(t)$ are coprime. Let $I = \langle g_1x_1 - f_1, \ldots, g_nx_n - f_n\rangle$. If $I \subset \mathbb{k}[t, x_1, \ldots, x_n]$, then $\boldsymbol{V}(I_1)$ is the smallest variety containing $F(\mathbb{k} \setminus W)$.

2. Plane curves

We present an application of Gröbner Bases theory for finding an polynomial equation for parametrically defined curve. Let us start with a well-known cubic.

Example 13
A plane curve studied by Descartes and Roberval in 1638, now called the *folium of Descartes*, highlighted the weaknesses of the method of Fermat in finding extreme of an algebraic curve. The rational parametrization of the Folium of Descartes is:

$$x = \tfrac{3t}{1+t^3}, \quad y = \tfrac{3t^2}{1+t^3}.$$

Note, that both pairs of the polynomials $3t$, $1+t^3$ and $3t^2$, $1+t^3$ (of one parameter) are coprime. Let $I = \langle f, g\rangle$, where

$$f := x + xt^3 - 3t, \quad g := y + yt^3 - 3t^2$$

be defined as in the Lemma 12, so the variety $\mathbf{V}(I_1)$ is the smallest variety containing $F(\mathbb{k} \setminus \mathbf{V}((1 + t^3)(1 + t^3)))$. Fixing lex order with $y \prec x \prec t$ and using *Singular* we get:

Computations in **Singular**:
```
> ring R = 0, (t, x, y), lp;
> ideal I = x + xt3 - 3t, y + yt3 - 3t2;
> eliminate (I, t);
_[1] x3 - 3xy + y3
```

And finally, we get the polynomial equation in two variables of the folium of Descartes:

$$x^3 - 3xy + y^3 = 0.$$

To find the polynomial equation of the folium of Descartes we can also use the

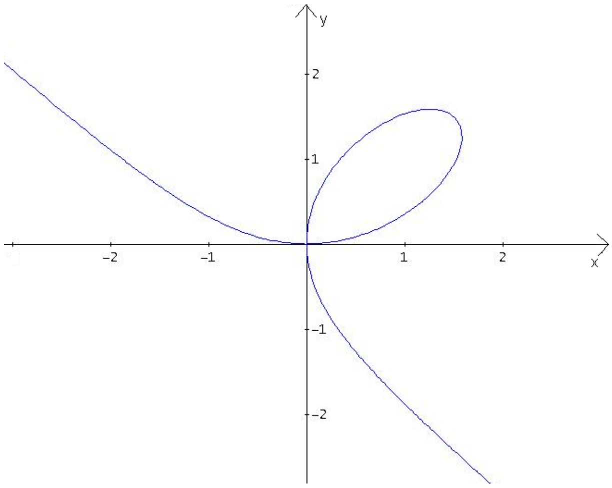

Figure 1: The folium of Descartes.

resultant - a classical tool of the theory of elimination. Resultant is the determinant of the Sylvester matrix for two fixed polynomials. For the definition and basic properties of the resultant see for Example [3]. Let us recall the fundamental theorem for applications of the resultant in the elimination theory:

Theorem 14
Let $f, g \in \mathbb{k}[x_1, \ldots, x_n]$ *and for fixed* $0 \leq i \leq 1$ *we have* $\deg_{x_i} f > 0$ *and* $\deg_{x_i} g > 0$. *Then*

$$\operatorname{Res}(f, g, x_i) \in \langle f, g \rangle \cap \mathbb{k}[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n].$$

Put $f := x + xt^3 - 3t$ and $g := y + yt^3 - 3t^2$. The resultant of f and g is:

$$\operatorname{Res}(f, g, t) = \det \begin{bmatrix} x & 0 & -3 & x & 0 & 0 \\ 0 & x & 0 & -3 & x & 0 \\ 0 & 0 & x & 0 & -3 & x \\ y & -3 & 0 & y & 0 & 0 \\ 0 & y & -3 & 0 & y & 0 \\ 0 & 0 & y & -3 & 0 & y \end{bmatrix} =$$

$$= -27(x^3 + y^3 - 3xy).$$

The equation

$$x^3 + y^3 - 3xy = 0,$$

is, by Theorem 14 and Theorem 8, the polynomial equation of the folium of Descartes.

Cubic curves play an important role in numerous area of mathematical and physical sciences. An interesting class of naturally parametrized cubics arises in Computer Aided Design (CAD). The underlying idea was developed in the late 1950s by two design engineers working for rival French car companies, namely Bézier (working for Renault) and Castlejau (working for Citröen) (see: [4], p. 8). A cubic Bézier's curve is a curve, defined by two end points and two intermediate points, so a cubic Bézier's curve interpolate end points and approximate intermediate points. Let $b_0,\ b_1,\ b_2,\ b_3$ be four point in the Euclidian plane, the curve starts at b_0 and ends in b_2. The cubic Bézier curve defined by control points $b_0,\ b_1,\ b_2,\ b_3$ is the path traced by the vector function:

$$B(t) = (1-t)^3 b_0 + 3t(1-t)^2 b_1 + 3t^2(1-t) b_3 + t^3 b_2.$$

This vector function gives the parametric equation of the cubic Bézier curve. We can use the presented effective method, based on the theory of Gröbner Basis, to find the polynomial equation for the cubic Bézier curve.

Example 15

Let $(-1,-1),\ (-1,1),\ (1,1),\ (1,-1)$ be the four points defining the cubic Bézier curve. For the polynomial equation of the curve defined by B, consider the parametrization of the Bézier curve:

$$\begin{aligned} x &= -4t^3 + 6t^2 - 1 \\ y &= 8t^3 - 12t^2 + 6t - 1 \end{aligned}$$

Fix lex order with $y \prec x \prec t$ and apply the *CoCoA* program for the Gröbner basis. We get:

Computations in **CoCoA**:

```
Use R ::= Q[t, x, y];
I := Ideal(x + 4t^3 − 6t^2 + 1, y − 8t^3 + 12t^2 − 6t + 1);
Elim([t], I);
Ideal(1/27x^3 + 1/18x^2y + 1/36xy^2 + 1/216y^3 − 1/8y)
```

The polynomial equation of the Bézier curve with control points $(-1,-1)$, $(-1,1)$, $(1,1)$, $(1,-1)$ is:

$$8x^3 + 12x^2y + 6xy^2 + y^3 - 27y = 0.$$

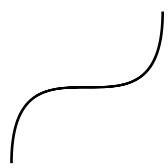

Figure 2: Bézier curve with control points $(-1,-1)$, $(1,1)$, $(-1,1)$, $(1,-1)$.

Proceed for the *resultant's methods* of determining the polynomial equation of the cubic Bézier curve we put

$$f := x + 4t^3 - 6t^2 + 1, \quad g := y - 8t^3 + 12t^2 - 6t + 1,$$

so the resultant of f and g is:

$$\mathrm{Res}(f, g, t) = \det \begin{bmatrix} 4 & -6 & 0 & x+1 & 0 & 0 \\ 0 & 4 & -6 & 0 & x+1 & 0 \\ 0 & 0 & 4 & -6 & 0 & x+1 \\ -8 & 12 & -6 & y+1 & 0 & 0 \\ 0 & -8 & 12 & -6 & y+1 & 0 \\ 0 & 0 & -8 & 12 & -6 & y+1 \end{bmatrix} =$$

$$= 64(8x^3 + 12x^2y + 6xy^2 + y^3 - 27y).$$

The equation

$$8x^3 + 12x^2y + 6xy^2 + y^3 - 27y = 0,$$

is, by Theorem 14 and Theorem 8, the polynomial equation of the considered Bézier curve.

3. Surfaces

Finally we consider parametrically defined surfaces. We use methods of the theory of Gröbner Bases to give an effective method for finding the implicit polynomial equation of the surface. First example is the following:

Example 16
Consider the surface given by the polynomial parametrization:

$$\begin{aligned} &x = t(u^2 - t^2), \\ &y = u, \\ &z = u^2 - t^2. \end{aligned}$$

We fix lex order with $z \prec y \prec x \prec t \prec u$ and compute the Gröbner basis of the polynomial ideal I generated by:

$$\begin{aligned} &g_1 = u - y \\ &g_2 = t^2 - y^2 + z \\ &g_3 = tz - x \\ &g_4 = tx - y^2z + z^2 \\ &g_5 = x^2 - y^2z^2 + z^3 \end{aligned}$$

The Elimination Theorem follows that $I_2 = I \cap \mathbb{R}[x,\ y,\ z] = \langle g_5 \rangle$, and thus by Theorem 10, $\mathbf{V}(g_5)$ solves the implicitization problem for the surface. The equation $x^2 - y^2z^2 + z^3 = 0$ defines the smallest variety in $\mathbb{C}^3$ containing the surface. But the question: 'If the surface fills up all of $\mathbf{V}(g_5) \subset \mathbb{R}^3$?' is still open. To answer, we must examine whether all partial solutions $(t,\ u,\ x,\ y,\ z) \in \mathbf{V}(I)$ extend to

$\mathbf{V}(g_5)$. Let us start with $(x,\ y,\ z) \in\ \mathbf{V}(I_2) = \mathbf{V}(g_5)$. The ideal I_2 is the first elimination ideal of I_1. By the Elimination Theorem, we have $I_1 = \langle g_2, \ldots,\ g_5\rangle$. Then Corollary 9 implies that (x, y, z) always extends to $(u,\ x,\ y, z) \in \mathbf{V}(I_1)$ since I_1 has a generator with a constant leading coefficient of u. And finally, from $(u,\ x,\ y, z) \in \mathbf{V}(I_1)$ to $(t,\ u,\ x,\ y,\ z) \in \mathbf{V}(I)$, using Corollary 9 again, we can always extend since $g_1 = u - y$ has a constant leading coefficient of t. We have thus proved that the considered surface is $\mathbf{V}(g_5)$.

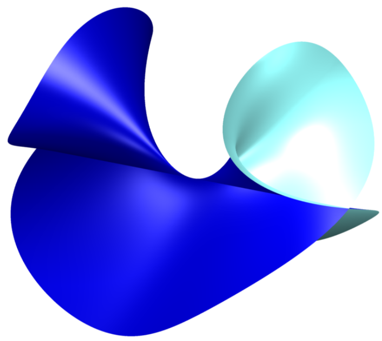

Figure 3: The locus of $x^2 - y^2z^2 + z^3 = 0$.

Example 17

Consider the surface given by the polynomial parametrization:

$$\begin{aligned} x &= uv \\ y &= u^2 \\ z &= v^2. \end{aligned}$$

We can proceed as in Example 16 or apply the function of the *Singular*. Let fix lex order with $z \prec y \prec x \prec v \prec u$ and compute the implicit polynomial equations:

Computations in **Singular**:

```
> ring R = 0, (u,v,x,y,z), lp;
> ideal I = x - uv, y - u2, z - v2;
> eliminate (I, uv);
_[1] x2 - yz
```

The surface is the locus of $x^2 - yz = 0$.

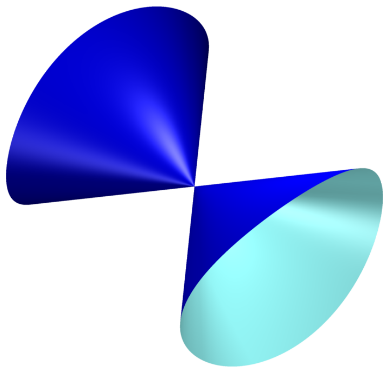

Figure 4: The locus of $x^2 - yz = 0$.

Example 18
Consider the surface given by the polynomial parametrization:

$$x = uv,$$
$$y = uv^2,$$
$$z = u^2.$$

We can proceed as in Example 16 or apply the *CoCoA*. For computation, we fix lex order with $z \prec y \prec x \prec v \prec u$ and find the implicit polynomial equation. The surface is the locus of $x^4 - y^2z = 0$.

Computations in **CoCoA**:

```
Use R ::=  Q[u, v, x, y, z];
I := Ideal(x − uv, y − uv^2, z − u^2);
Elim([u, v], I);
Ideal(x^4 − y^2z)
```

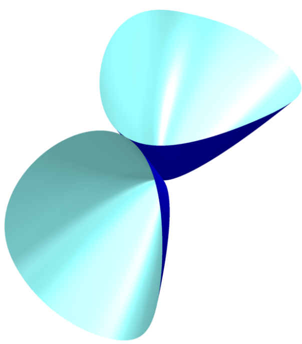

Figure 5: The locus of $x^4 - y^2z = 0$.

Finally we consider the *Enneper surface*,

Example 19
The parametrical equations of Enneper surface are:

$$x = 3u + 3uv^2 - u^3,$$
$$y = 3v + 3u^2v - v^3,$$
$$z = 3u^2 - 3v^2.$$

For finding the polynomial equation of the smallest variety V containing the Enneper surface we consider the ideal $I = \langle x - 3u - 3uv^2 + u^3,\ y - 3v - 3u^2v + v^3,\ z - 3u^2 + 3v^2 \rangle$. Fix lex order with $z \prec y \prec x \prec v \prec u$ and compute the Gröbner basis for I. By the Elimination Theorem $I_2 = I \cap \mathbb{R}[x,\ y,\ z]$ is the ideal generated by first polynomial of the Gröbner basis calculated in the *Singular*. Using similar argument as in Example 16 we can show that this polynomial defines the smallest subvariety in $\mathbb{C}^3$ in which the Enneper surface is contained.

Figure 6: The Enneper surface.

Computations in **Singular**:

```
> ring R = 0, (u, v, x, y, z), lp; option(redSB);
> ideal I = x − 3u − 3uv2 + u3, y − 3v − 3u2v + v3, z − 3u2 + 3v2;
> std(I);
_[1] = 19683x6 − 59049x4y2 + 10935x4z3 + 118098x4z2 − 59049x4z+
59049x2y4 + 56862x2y2z3 + 118098x2y2z + 1296x2z6 + 34992x2z5+
174960x2z4 − 314928x2z3 − 19683y6 + 10935y4z3 − 118098y4z2−
59049y4z − 1296y2z6 + 34992y2z5 − 174960y2z4 − 314928y2z3 − 64z9+
10368z7 − 419904z5
_[2] = 8748vy3z2 + 648vyz5 + 5832vyz4 + 17496vyz3 + 17496vyz2 − 729x4z−
2187x4 + 5832x2y2z + 4374x2y2 − 189x2z4 − 2997x2z3 − 5103x2z2 + 6561x2z−
5103y4z − 2187y4 − 945y2z4 + 81y2z3 − 16767y2z2 − 6561y2z + 8z7−
48z6 − 864z5 + 3888z4 + 17496z3
_[3] = 27vx2z + 81vx2 + 135vy2z − 81vy2 + 8vz4 + 96vz3 + 216vz2+
81x2y − 81y3 − 12yz3 − 324yz
_[4] = 4374vx2y + 8748vy3z − 4374vy3 + 648vyz4 + 5184vyz3 + 17496vyz2−
729x4 + 5832x2y2 − 189x2z3 − 2430x2z2 + 2187x2z − 5103y4 − 945y2z3+
972y2z2 − 19683y2z + 8z6 − 72z5 − 648z4 + 5832z3
_[5] = 2187vx4 + 69984vx2 + 8748vy4z − 2187vy4 + 648vy2z4 + 3240vy2z3−
11664vy2z2 + 139968vy2z − 69984vy2 − 192vz6 − 3456vz5 − 15552vz4+
20736vz3 + 186624vz2 − 729x4y + 5832x2y3 − 189x2yz3 − 7047x2yz2−
23328x2yz + 69984x2y − 5103y5 − 945y3z3 + 1215y3z2 + 5832y3z − 69984y3+
8yz6 + 288yz5 + 216yz4 + 5184yz3 + 93312yz2 − 279936yz
_[6] = 18v2z2 + 54v2z − 54vyz − 27x2 + 27y2 + z3 − 18z2 + 81z
_[7] = 54v2yz + 27vx2 − 27vy2 + 8vz3 + 72vz2 − 9yz2 − 27yz
_[8] = 243v2x2 − 243v2y2 − 1296v2z − 108vyz2 + 324vyz + 108x2z + 648x2+
135y2z − 648y2 − 4z4 + 48z3 + 108z2 − 1944z
_[9] = 2v3 + vz + 3v − y
_[10] = 27uy2 − 8uz3 + 72uz2 − 36v2xz − 27vxy − 24xz2
_[11] = 9ux + 6v2z − 9vy + z2 − 9z
_[12] = 4uvz − 3uy + 3vx
_[13] = 9uvy − 2uz2 + 18uz − 9v2x − 6xz
_[14] = 6uv2 − uz + 9u − 3x
_[15] = 3u2 − 3v2 − z
```

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