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Conjugations in $\mathbb{C}^n$

Streszczenie. W pracy bedziemy rozpatrywać izometryczne, antyliniowe inwolucje w przestrzeni $\mathbb{C}^n$. W szczególności skoncentrujemy sie na odwzorowaniach liniowych, "symetrycznych" wzgledem zadanych inwolucji. Jednym z naszych głównych celów bedzie zbadanie zachowania macierzy Jordana wzgledem wybranych inwolucji.

Abstract. In this paper we study cojugations (isometric antilinear involutions) in $\mathbb{C}^n$. In particular we concentriate on linear mappings which are "symmetric" with respect to conjugations. One of our aims is to investigate the behaviour of Jordan matrices according to various conjugations.

1. Introduction

The mapping $C\colon z \mapsto \bar{z}$, $z \in \mathbb{C}$, is not linear because it does not satisfy the condition of homogeneity. This kind of mapping can be generalized to all complex Hilbert spaces and, in particular, to specific subspaces of analytic functions on the unit disc in the context of truncated Toeplitz operators. In [1], [2], [3], [4], [6], [7] these mappings have been considered in spaces of analytic functions. Our aim is to consider these mappings only on $\mathbb{C}^n$.

2. Basic definitions and examples

Let us consider $\mathbb{C}^n$ with an inner product $\langle \cdot, \cdot \rangle$ and let I denote the identity mapping in $\mathbb{C}^n$.

Definition 2.1 ([5, Definition 1.1])
A *conjugation* $C\colon \mathbb{C}^n \to \mathbb{C}^n$ is an isometric involution, i.e., mapping C satisfies:

(i) $\langle z, w \rangle = \langle Cw, Cz \rangle$ for all $z, w \in \mathbb{C}^n$,

(ii) $C^2 = I$.

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By this definition we have the following

Corollary 2.2
Let $C\colon \mathbb{C}^n \to \mathbb{C}^n$ be a conjugation in $\mathbb{C}^n$. Then C is antylinear, i.e.,

$$C(\alpha z + \beta w) = \bar{\alpha}C(z) + \bar{\beta}C(w),$$

for all $z, w \in \mathbb{C}^n$ and $\alpha, \beta \in \mathbb{C}$.

Proof. The standard properties of inner product show that

$$\begin{aligned}
&||C(\alpha z+\beta w) - \bar{\alpha}Cz - \bar{\beta}Cw||^2 \\
&= \langle C(\alpha z+\beta w) - \bar{\alpha}Cz - \bar{\beta}Cw, C(\alpha z+\beta w) - \bar{\alpha}Cz - \bar{\beta}Cw\rangle \\
&= \langle C(\alpha z+\beta w), C(\alpha z+\beta w)\rangle + \langle C(\alpha z+\beta w), -\bar{\alpha}Cz\rangle + \langle C(\alpha z+\beta w), -\bar{\beta}Cw\rangle \\
&\qquad + \langle -\bar{\alpha}Cz, C(\alpha z+\beta w)\rangle + \langle -\bar{\alpha}Cz, -\bar{\alpha}Cz\rangle + \langle -\bar{\alpha}Cz, -\bar{\beta}Cw\rangle \\
&\qquad + \langle -\bar{\beta}Cw, C(\alpha z+\beta w)\rangle + \langle -\bar{\beta}Cw, -\bar{\alpha}Cz\rangle + \langle -\bar{\beta}Cw, -\bar{\beta}Cw\rangle \\
&= \alpha\bar{\alpha}\langle z,z\rangle + \beta\bar{\alpha}\langle w,z\rangle + \alpha\bar{\beta}\langle z,w\rangle + \beta\bar{\beta}\langle w,w\rangle - \alpha\bar{\alpha}\langle z,z\rangle - \alpha\bar{\beta}\langle z,w\rangle \\
&\qquad - \bar{\alpha}\beta\langle w,z\rangle - \beta\bar{\beta}\langle w,w\rangle - \alpha\bar{\alpha}\langle z,z\rangle - \bar{\alpha}\beta\langle w,z\rangle + \alpha\bar{\alpha}\langle z,z\rangle + \bar{\alpha}\beta\langle w,z\rangle \\
&\qquad - \alpha\bar{\beta}\langle z,w\rangle - \beta\bar{\beta}\langle w,w\rangle + \alpha\bar{\beta}\langle z,w\rangle + \beta\bar{\beta}\langle w,w\rangle = 0.
\end{aligned}$$

■

Consider various examples of conjugations.

Example 2.3
(1) $C\colon \mathbb{C}^n \to \mathbb{C}^n$, $C(z_1, z_2, \ldots, z_n) = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$ named *standard conjugation* (see [2, Prealiminaries 2.1]);

(2) $C\colon \mathbb{C}^n \to \mathbb{C}^n$, $C(z_1, z_2, \ldots, z_n) = (\bar{z}_n, \ldots, \bar{z}_2, \bar{z}_1)$ named *canonical conjugation* (see [2, Example 4];

(3) $C\colon \mathbb{C}^2 \to \mathbb{C}^2$, $C(z_1, z_2) = \left(\frac{1}{\sqrt{1+a^2}}\bar{z}_1 + \frac{a}{\sqrt{1+a^2}}\bar{z}_2, \frac{a}{\sqrt{1+a^2}}\bar{z}_1 - \frac{1}{\sqrt{1+a^2}}\bar{z}_2\right)$, $a \in \mathbb{C} \setminus \{i, -i\}$ (see [2, Example 5]);

(4) $C\colon \mathbb{C}^n \to \mathbb{C}^n$, $C(z_1, \ldots, z_k, z_{k+1}, \ldots, z_n) = (\bar{z}_k, \ldots, \bar{z}_1, \bar{z}_n \ldots, \bar{z}_{k+1})$;

(5) $C\colon \mathbb{C}^8 \to \mathbb{C}^8$, $C(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) = (\bar{z}_2, \bar{z}_1, \bar{z}_5, \bar{z}_4, \bar{z}_3, \bar{z}_8, \bar{z}_7, \bar{z}_6)$.

3. C–symmetric linear mappings

Before stateing the main definition, recall some basic properties of linear mappings. Let $L(\mathbb{C}^n)$ be the set of all linear mappings in $\mathbb{C}^n$ and let $M_{n\times n}(\mathbb{C})$ be the set of all n–by–n matrices. Let us fix $\mathcal{B} = \{e_1, \ldots, e_n\}$ an orthonormal basis in $\mathbb{C}^n$. It is well known that there exists an isomorphism

$$L(\mathbb{C}^n) \simeq M_{n\times n}(\mathbb{C}).$$

In particular, for $A \in L(\mathbb{C}^n)$,

$$A \simeq [a_{kl}] = [\langle Ae_l, e_k\rangle]_{\substack{k=1,\dots,n\\ l=1,\dots,n}} = \begin{bmatrix} \langle Ae_1, e_1\rangle & \langle Ae_2, e_1\rangle & \cdots & \langle Ae_n, e_1\rangle \\ \langle Ae_1, e_2\rangle & \langle Ae_2, e_2\rangle & & \langle Ae_n, e_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle Ae_1, e_n\rangle & \langle Ae_2, e_n\rangle & \cdots & \langle Ae_n, e_n\rangle \end{bmatrix}.$$

Recall also that for every $A \in L(\mathbb{C}^n)$ there exists the adjoint mapping which is defined by the equality

$$\langle Az, w\rangle = \langle z, A^* w\rangle, \quad z, w \in \mathbb{C}^n.$$

Note that if $A = [a_{kl}]$, then $A^* = [\bar{a}_{kl}]^T$.

Definition 3.1
Let $C\colon \mathbb{C}^n \to \mathbb{C}^n$ be a conjugation. For $A \in L(\mathbb{C}^n)$ recall that

(i) A is said to be *C–symmetric* if $CAC = A^*$, [2, p.1286];

(ii) A is *C–skew-symmetric* if $CAC = -A^*$, [7, p.13].

Proposition 3.2
Let $C\colon \mathbb{C}^n \to \mathbb{C}^n$ be a conjugation, $A, B \in L(\mathbb{C}^n)$ and let $\alpha, \beta \in \mathbb{C}$. The following hold.

(i) *A is C–symmetric if and only if $\langle Az, Cw\rangle = \langle z, CAw\rangle$, $z, w \in \mathbb{C}^n$;*

(ii) *A is C–skew-symmetric if and only if $\langle Az, Cw\rangle = \langle z, -CAw\rangle$, $z, w \in \mathbb{C}^n$;*

(iii) *The identity I is C–symmetric;*

(iv) *The mappings A, B are C–symmetric if and only if $\alpha A + \beta B$ is C–symmetric for all $\alpha, \beta \in \mathbb{C}$.*

Proof. We only prove (iv). By the definition we have $CAC = A^*$ and $CBC = B^*$. Easy calculation shows that

$$\begin{aligned} C(\alpha A + \beta B)C &= C(\alpha A)C + C(\beta B)C = \bar{\alpha} CAC + \bar{\beta} CBC \\ &= \bar{\alpha} A^* + \bar{\beta} B^* = (\alpha A + \beta B)^*. \end{aligned}$$

■

We now give examples of C–symmetric mappings.

Example 3.3 ([2, Example 5])
Let us take a conjugation $C_a\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$C_a(z_1, z_2) = \left(\tfrac{1}{\sqrt{1+a^2}} \bar{z}_1 + \tfrac{a}{\sqrt{1+a^2}} \bar{z}_2, \tfrac{a}{\sqrt{1+a^2}} \bar{z}_1 - \tfrac{1}{\sqrt{1+a^2}} \bar{z}_2 \right), \quad a \in \mathbb{C} \setminus \{i, -i\}.$$

The linear mapping $A \in L(\mathbb{C}^2)$ given by $A(z_1, z_2) = (z_1 + a z_2, 0)$ is C_a–symmetric.

Example 3.4
Let us consider $\mathbb{C}^3$ and the canonical conjugation [Example 2.3(2)] in this space, i.e., $C(z_1, z_2, z_3) = (\bar{z}_3, \bar{z}_2, \bar{z}_1)$ for all $(z_1, z_2, z_3) \in \mathbb{C}^3$. Let $A \in L(\mathbb{C}^3)$ be given by $A\big(z_1, z_2, z_3\big) = (iz_1 - iz_2 + 2iz_3, -iz_2 - iz_3, iz_3)$. It can be seen that A is C–symmetric by the following:

$$\begin{aligned} CAC(z_1, z_2, z_3) &= CA(\bar{z}_3, \bar{z}_2, \bar{z}_1) \\ &= C(i\bar{z}_3 - i\bar{z}_2 + 2i\bar{z}_1, -i\bar{z}_2 - i\bar{z}_1, i\bar{z}_1) \\ &= (-iz_1, iz_1 + iz_2, -2iz_1 + iz_2 - iz_3) \\ &= A^*(z_1, z_2, z_3). \end{aligned}$$

4. Special properties of Jordan Matrices

It what follows we will see that C–symmetry of a linear mapping in $\mathbb{C}^n$ can be denoted from its matrix representation. Let us consider the canonical conjugation [Example 2.3(2)] and the linear mapping $A \in L(\mathbb{C}^3)$ given by

$$A(z_1, z_2, z_3) = (iz_1 - iz_2 + 2iz_3, -iz_2 - iz_3, iz_3).$$

We can observe that the matrix representation $[a_{i,j}]_{\substack{i=1,2,3\\ j=1,2,3}}$ of this mapping, according to the canonical basis, is symmetric with respect to its second diagonal, i.e.,

$$a_{11} = a_{33}, \quad a_{21} = a_{32}, \quad a_{12} = a_{23}.$$

We will show below that every linear mapping $A \in L(\mathbb{C}^n)$, which has this property, is C–symmetric.

Theorem 4.1
Let C be the canonical conjugation in $\mathbb{C}^n$ [Example 2.3(2)] *and let $\mathcal{B} = \{e_1, \dots, e_n\}$ be the canonical basis. Then $A \in L(\mathbb{C}^n)$ is C–symmetric if and only if its matrix representation $[a_{ij}]$ accoording to the basis $\mathcal{B}$ is symmetric with respect to the second diagonal, i.e., $a_{ij} = a_{n-j+1,n-i+1}$, $i, j = 1, 2, \dots, n$.*

Proof. Firstly let us observe that $Ce_i = e_{n-i+1}$, $i = 1, 2, ..., n$. Using only the definition of a conjugation we get

$$\begin{aligned} a_{ij} &= \langle Ae_j, e_i\rangle = \langle Ce_i, CAe_j\rangle = \langle Ce_i, A^*Ce_j\rangle \\ &= \langle ACe_i, Ce_j\rangle = \langle Ae_{n-i+1}, e_{n-j+1}\rangle = a_{n-j+1,n-i+1}. \end{aligned}$$

■

In what follows $\mathbb{C}^n$ is decomposed to the orthogonal sum, i.e. $w_1 + \dots + w_k = n$ and $\mathbb{C}^n = \mathbb{C}^{w_1} \oplus \mathbb{C}^{w_2} \oplus \dots \oplus \mathbb{C}^{w_k}$. We can also consider a conjugation $C\colon \mathbb{C}^n \to \mathbb{C}^n$ given by the orthogonal sum of canonical conjugations C_{w_i} on each $\mathbb{C}^{w_i}$, $i = 1, \dots, k$, i.e.,

$$C = C_{w_1} \oplus C_{w_2} \oplus \dots \oplus C_{w_k}, \tag{4.1}$$

that means

$$C(z_1,\dots,z_{w_1},z_{w_1+1},\dots,z_{w_2},\dots,z_{w_{k-1}},\dots,z_{w_k}) \\ = (\bar{z}_{w_1},\dots,\bar{z}_1,\bar{z}_{w_2},\dots,\bar{z}_{w_1+1},\dots,\bar{z}_{w_k},\dots,\bar{z}_{w_k-1}).$$

Now let us consider a Jordan block given by

$$J_m(\lambda) = \lambda I + S_m, \quad \lambda \in \mathbb{C}.$$

By S_m we mean the trunkated shift in $\mathbb{C}^n$, $S_m(z_1,\dots,z_m) = (0,z_1,\dots,z_{m-1})$. Then $S_m^*(z_1,\dots,z_m) = (z_2,z_3,\dots,z_m,0)$. Note that S_m is C–symmetric according to the canonical conjugation [Example 2.3(2)]. Indeed, for every $(z_1,\dots,z_m) \in \mathbb{C}^m$ we have

$$CS_mC(z_1,\dots,z_m) = CS_m(\bar{z}_m,\dots,\bar{z}_1) = C(0,\bar{z}_m,\dots,\bar{z}_2) \\ = (z_2,\dots,z_m,0) = S_m^*(z_1,\dots,z_m).$$

Note that the Jordan block $J_m(\lambda)$ is C–symmetric with respect to the canonical conjugation by Proposition 3.2 for all $\lambda \in \mathbb{C}$.

The well-known cannonical Jordan decomposition theorem says that for every linear mapping $A \in L(\mathbb{C}^n)$ there exists an invertable mappipng V such that its matrix representation is

$$V^{-1}AV = J = \begin{pmatrix} J_{w_1}(\lambda_1) & 0 & & \cdots & 0 \\ 0 & J_{w_2}(\lambda_2) & & & \vdots \\ \vdots & & \ddots & & \\ & & & J_{w_{k-1}}(\lambda_{k-1}) & 0 \\ 0 & \cdots & & 0 & J_{w_k}(\lambda_k) \end{pmatrix},$$

where $\lambda_m \in \mathbb{C}, m = 1,\dots,k$ and $w_1 + \cdots + w_k = n$. We can see that the Jordan matrix J is usually not C–symmetric according to the canonical conjugation in C^n. Now let us consider the conjugation $C\colon \mathbb{C}^n \to \mathbb{C}^n$ given by equation (4.1).

We can notice that Jordan matrix is C–symmetric and we show this property by following calculation:

$$CJC = (C_{w_1} \oplus C_{w_2} \oplus \cdots \oplus C_{w_k})(J_{w_1} \oplus J_{w_2} \oplus \cdots \oplus J_{w_k})(C_{w_1} \oplus C_{w_2} \oplus \cdots \oplus C_{w_k}) \\ = C_1J_{w_1}C_1 \oplus C_2J_{w_2}C_2 \oplus \cdots \oplus C_kJ_{w_k}C_k = J_{w_1}^* \oplus J_{w_2}^* \oplus \cdots \oplus J_{w_k}^* = J^*.$$

5. Some basic properties of C–symmetric mappings

In this section we will study some basic properties of C–symmetric mappings in $\mathbb{C}^n$.

Theorem 5.1 ([2, Proposition 1])
Let C be a conjugation in $\mathbb{C}^n$ and let $A \in L(\mathbb{C}^n)$ be invertible. The mapping A is C–symmetric if and only if A^{-1} is also C–symmetric, i.e., $CA^{-1}C = (A^{-1})^*$.

Proof. Note by (2.1) that the conjugation C is invertible as a mapping with $C^{-1} = C$. Standard properties of inverse mappings show that

$$CA^{-1}C = C^{-1}A^{-1}C^{-1} = (CAC)^{-1} = (A^*)^{-1} = (A^{-1})^*.$$

■

THEOREM 5.2
Let C be a conjugation in $\mathbb{C}^n$ and $A \in L(\mathbb{C}^n)$. The mapping A is C–symmetric and C–skew-symmetric if and only if $A \equiv 0$.

Proof. If we can write $CAC = A^*$ and $CAC = -A^*$, then it implies $2A^* = 0$, which means $A \equiv 0$. The converse is obvious. ■

We can also introduce orthogonality in connection with C–symmetry. Define the following relation between vectors. Let C be a conjugation in $\mathbb{C}^n$. We follow the notation of [7, p.14] and define $[z, w]_C := \langle z, Cw\rangle$. Vectors z and w are said to be *C–orthogonal*, if

$$[z, w]_C = 0.$$

Recall without proof that the following holds.

THEOREM 5.3 ([7, Proposition 2.1])
Let $A \in L(\mathbb{C}^n)$ be a C–symmetric mapping in $\mathbb{C}^n$. Eigenvectors of the mapping A, which correspond to different eigenvalues, are C–orthogonal.

EXAMPLE 5.4 ([7, Example 2.4])
Let C be the canonical conjugation in $\mathbb{C}^2$ (2.3), and let $A \in L(\mathbb{C}^n)$ be a C–symmetric mapping given by $A(z_1, z_2) = (z_1 + iz_2, iz_1)$. The eigenvalues of A are $\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ and the corresponding eigenvectors are $v_1 = \frac{1}{2\sqrt{2}}(\sqrt{3} - i, 2)$, $v_2 = \frac{1}{2\sqrt{2}}(-\sqrt{3} - i, 2)$. Vectors v_1 i v_2 are not orthogonal but they are C–orthogonal.

THEOREM 5.5 ([5, p. 2758])
Let $C\colon \mathbb{C}^n \to \mathbb{C}^n$ be a conjugation. Every linear mapping $A \in L(\mathbb{C}^n)$ can be represented by

$$A = A_C + A_{SC},$$

where $A_C = \frac{1}{2}(A + CA^*C)$ and $A_{SC} = \frac{1}{2}(A - CA^*C)$. The mapping A_C is C–symmetric and A_{SC} is C–skew-symmetric. That decomposition is unique.

Proof. Let $z, w \in \mathbb{C}^n$. Using only the definition of a conjugation we show below that $(CAC)^* = CA^*C$ [4, proof of Theorem 4.2]. Namely,

$$\langle CACz, w\rangle = \langle Cw, ACz\rangle = \langle A^*Cw, Cz\rangle = \langle z, CA^*Cw\rangle = \langle (CA^*C)^*z, w\rangle.$$

Now we will show that A_C and A_{SC} are C–symmetric and C–skew-symmetric, respectively. Indeed,

$$\begin{aligned} A_C^* &= \tfrac{1}{2}(A + CA^*C)^* = \tfrac{1}{2}A^* + \tfrac{1}{2}(CA^*C)^* = \tfrac{1}{2}A^* + \tfrac{1}{2}C(A^*)^*C \\ &= \tfrac{1}{2}CCA^*CC + \tfrac{1}{2}CAC = C\tfrac{1}{2}(CA^*C + A)C = CA_CC \end{aligned}$$

and

$$\begin{aligned} A_{SC}^* &= \tfrac{1}{2}(A - CA^*C)^* = \tfrac{1}{2}A^* - \tfrac{1}{2}(CA^*C)^* = \tfrac{1}{2}A^* - \tfrac{1}{2}C(A^*)^*C \\ &= \tfrac{1}{2}CCA^*CC - \tfrac{1}{2}CAC = C\tfrac{1}{2}(CA^*C - A)C = -C\tfrac{1}{2}(A - CA^*C)C \\ &= -CA_{SC}C. \end{aligned}$$

Note that

$$\tfrac{1}{2}(A + CA^*C) + \tfrac{1}{2}(A - CA^*C) = \tfrac{1}{2}A + \tfrac{1}{2}A + \tfrac{1}{2}CA^*C - \tfrac{1}{2}CA^*C = A.$$

Now we proof that this decomposition is unique. Let us suppose that $A = A_{1C} + A_{1SC}$ and $A = A_{2C} + A_{2SC}$. Then $0 = A_{1C} - A_{2C} + A_{1SC} - A_{2SC}$. Therefore $A_{1C} - A_{2C} = A_{2SC} - A_{1SC}$. Hence by Theorem 5.2 we have $A_{1C} - A_{2C} = 0$ and $A_{2SC} - A_{1SC} = 0$, which implies $A_{1C} = A_{2C}$ and $A_{1SC} = A_{2SC}$. ■

Example 5.6
Consider $\mathbb{C}^3$, the canonical conjugation $C\colon \mathbb{C}^3 \to \mathbb{C}^3$, $C(z_1, z_2, z_3) = (\bar{z}_3, \bar{z}_2, \bar{z}_1)$ and $A \in L(\mathbb{C}^3)$ given by $A(z_1, z_2, z_3) = (iz_1 + z_2 + iz_3, -iz_2 + z_3, z_1 + iz_3)$. If $A_C = \frac{1}{2}(A + CA^*C)$ and $A_{SC} = \frac{1}{2}(A - CA^*C)$, then $A = A_C + A_{SC}$.

$$\begin{aligned} A_C(z_1, z_2, z_3) &= \tfrac{1}{2}((iz_1 + z_2 + iz_3, -iz_2 + z_3, z_1 + iz_3) + CA^*(\bar{z_3}, \bar{z_2}, \bar{z_1})) \\ &= \tfrac{1}{2}((iz_1 + z_2 + iz_3, -iz_2 + z_3, z_1 + iz_3) \\ &\quad + C(-i\bar{z_3} + \bar{z_1}, \bar{z_3} + i\bar{z_2}, -i\bar{z_3} + \bar{z_2} - i\bar{z_1})) \\ &= \tfrac{1}{2}((iz_1 + z_2 + iz_3, -iz_2 + z_3, z_1 + iz_3) \\ &\quad + (iz_1 + z_2 + iz_3, -iz_2 + z_3, z_1 + iz_3)) \\ &= (iz_1 + z_2 + iz_3, -iz_2 + z_3, z_1 + iz_3) \end{aligned}$$

$$\begin{aligned} A_S(z_1, z_2, z_3) &= \tfrac{1}{2}((iz_1 + z_2 + iz_3, -iz_2 + z_3, z_1 + iz_3) - CA^*(\bar{z_3}, \bar{z_2}, \bar{z_1})) \\ &= \tfrac{1}{2}((iz_1 + z_2 + iz_3, -iz_2 + z_3, z_1 + iz_3) \\ &\quad - C(-i\bar{z_3} + \bar{z_1}, \bar{z_3} + i\bar{z_2}, -i\bar{z_3} + \bar{z_2} - i\bar{z_1})) \\ &= \tfrac{1}{2}((iz_1 + z_2 + iz_3, -iz_2 + z_3, z_1 + iz_3) \\ &\quad + (iz_1 + z_2 + iz_3, -iz_2 + z_3, z_1 + iz_3)) = 0 \end{aligned}$$

$$\begin{aligned} (A_C + A_S)(z_1, z_2, z_3) &= ((iz_1 + z_2 + iz_3, -iz_2 + z_3, z_1 + iz_3) + 0) \\ &= A(z_1, z_2, z_3) \end{aligned}$$

The proof that A_C is C–symmetric and A_{SC} is C–skew-symmetric we left to the reader.

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