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Invariant subspaces, reflexivity, hyperreflexivity

Abstract. The problem of existence of a nontrivial invariant subspace for an arbitrary operator in a complex Hilbert space is one of the most challenging unsolved problem in the whole operator theory. The problem and partial results are presented. The notions of the reflexivity and the hyperreflexivity are closely related to the main one.

1. Introduction

The paper is based on the talk with the same title given on „XI Sympozjum Kół Naukowych; Obudź w sobie matematykę!", Kraków, 08-10.04.2016. The aim of the talk and of the paper is to present one of the modern topics in Mathematics: the operator theory in Hilbert spaces. Especially, almost a hundred years old problem of the existence of a nontrivial invariant subspace for an arbitrary operator in a complex Hilbert space. We also discuss the reflexivity and the hyperreflexivity problem as closely connected with the main one. Some open problems are presented.

The talk and the paper are dedicated to students, also to undergraduate. The author have tried to present the topic in a simple way, giving rather ideas and examples than precise proofs.

2. Hilbert spaces and operators

In what follows we will consider a *complex separable Hilbert space* $\mathcal{H}$. It means that $\mathcal{H}$ is a vector space over $\mathbb{C}$ and there is a complex inner product $\langle\cdot,\cdot\rangle\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$. The inner product defines a norm $\|\cdot\|\colon \mathcal{H}\to\mathbb{R}_+$, $\|h\|=\sqrt{\langle h,h\rangle}$, for $h\in\mathcal{H}$, and a metric (distance) $d(\cdot,\cdot)\colon \mathcal{H}\times\mathcal{H}\to\mathbb{R}_+$, $d(f,g)=\|f-g\|=\sqrt{\langle f-g,f-g\rangle}$ for $f,g\in\mathcal{H}$. According to the definition of a Hilbert space we assume that the space $(\mathcal{H},d)$ as a metric space is complete. The set $\{e_k\}_{k\in\mathcal{I}}\subset\mathcal{H}$ is called an *orthonormal basis* of $\mathcal{H}$ if the set $\{e_k\}_{k\in\mathcal{I}}$ consist of orthogonal unit vectors (i.e.

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for all $k, m \in \mathcal{I}$ we have $\langle e_k, e_m \rangle = 0$ for $k \neq m$ and $\|e_k\| = 1$) and the set $\{e_k\}_{k\in\mathcal{I}}$ is maximal in the sense that there is no set of orthogonal unit vectors properly containing the set $\{e_k\}_{k\in\mathcal{I}}$. The essential property of the basis is that each $h \in \mathcal{H}$ can be uniquely represented as $h = \sum_{k\in\mathcal{H}} h_k e_k$ for some set of complex numbers $\{h_k\}$. For more properties of Hilbert spaces see [9].

Let as recall some examples of Hilbert spaces.

Example 2.1
The space $\mathbb{C}^n$ with the inner product:

$$\langle z, w\rangle = z_1\overline{w}_1 + z_2\overline{w}_2 + \cdots + z_n\overline{w}_n = \sum_{k=1}^{n} z_k\overline{w}_k$$

for $z, w \in \mathbb{C}^n$, $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n)$, is a Hilbert space.

Example 2.2
The space $\mathbb{C}^n$ or any finite dimensional complex vector space V ($\dim V < \infty$) with an arbitrary inner product is a complex Hilbert space.

Example 2.3 ([9, Example I.1.3])
The space of complex sequences indexed by $\mathbb{N}$:

$$l^2_+ = \{(h_k)_{k=0}^{\infty} \subset \mathbb{C} : \sum_{k=0}^{\infty} |h_k|^2 < \infty\}$$

with the inner product: $\langle h, g\rangle = \sum\limits_{k=0}^{\infty} h_k\overline{g_k}$ for $h = (h_k), g = (g_k) \in l^2_+$.

Example 2.4 ([9, Example I.1.7])
The space of complex sequences indexed by all integers:

$$l^2 = \{(h_k)_{k=-\infty}^{+\infty} \subset \mathbb{C} : \sum_{k=-\infty}^{+\infty} |h_k|^2 < \infty\}$$

with an inner product: $\langle h, g\rangle = \sum\limits_{k=-\infty}^{+\infty} h_k\overline{g_k}$ for $h = (h_k), g = (g_k) \in l^2$.

Example 2.5 ([9, Example I.1.3])
A space of measurable L^2 functions

$$L^2[-\pi,\pi] = \{f\colon [-\pi,\pi] \to \mathbb{C} \text{ measurable} : \int\limits_{[-\pi,\pi]} |f(t)|^2\, dt < \infty\},$$

with an inner product $\langle f, g\rangle = \frac{1}{2\pi} \int\limits_{[-\pi,\pi]} f(t)\overline{g(t)}\, dt$.

The space $L^2[-\pi,\pi]$ is unitarily equivalent to l^2 space and this equivalence is given by the Fourier representation $h = \sum_{n=-\infty}^{+\infty} h_n e^{in\theta} \to (h_n)_{n=-\infty}^{+\infty}$, see [9, Theorem I.5.7].

Example 2.6 ([13])
The Hardy space:

$$H^2 = \{h\colon [-\pi,\pi] \to \mathbb{C}\colon h(\theta) = \sum_{n=0}^{\infty} h_n e^{in\theta}, \int\limits_{[-\pi,\pi]} |h(\theta)|^2\, d\theta < \infty\}.$$

As we see H^2 can be seen as a subspace of $L^2[-\pi,\pi]$ and using the Fourier representation the space H^2 can be identified with l^2_+.

The space H^2 can be also seen as a subspace of the set of all holomorphic functions on the unit disc $\mathbb{D}$

$$H^2 = \{h\colon \mathbb{D} \to \mathbb{C} : h(z) = \sum_{n=0}^{\infty} h_n z^n, \sup_{0<r<1} \int |h(re^{i\theta})|^2\, d\theta < \infty\}.$$

Moreover, functions from H^2 have radial limits at almost every point in $\partial\mathbb{D}$ the boundary of $\mathbb{D}$.

Example 2.7 ([20, Chap. II, §9])
The space of states of a given system of particles in quantum mechanics.

Example 2.8 ([28, Chap. 3],[27, Chap. 3])
The space generated by a discrete stochastic process $\{\xi_n\}_{n=0}^{\infty} \subset l^2_+$ or continuous stochastic process $\{\xi_t\}_{t\geqslant 0} \subset L^2\left(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt\right)$, where

$$L^2\left(\mathbb{R}, \tfrac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt\right) = \{f\colon \mathbb{R} \to \mathbb{R} \text{ measurable} : \tfrac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} |f(t)|^2\, e^{-\frac{t^2}{2}}\, dt < \infty\}.$$

A linear function $T\colon \mathcal{H} \to \mathcal{H}$ is called a *bounded operator* on $\mathcal{H}$, if there is M such that $\|Th\| \leqslant M\|h\|$ for all $h \in \mathcal{H}$. In fact, boundedness of the operator T is equivalent its continuity. Let $L(\mathcal{H})$ denote the set of all bounded linear operators. $L(\mathcal{H})$ is a vector space. Moreover, it is an algebra with a composition as a multiplication. The space $L(\mathcal{H})$ is equipped with the norm $\|\cdot\|\colon L(\mathcal{H}) \to \mathbb{R}_+$, $\|T\| = \sup\{\|Tx\| : x \in \mathcal{H},\ \|x\| \leqslant 1\}$ for $T \in L(\mathcal{H})$. The norm defines the topology in $L(\mathcal{H})$. There is also Weak Operator Topology (WOT) on $L(\mathcal{H})$, i.e. a topology given by seminorms $(h,g) \to \langle Th, g\rangle, T \in L(\mathcal{H})$ for $h, g \in \mathcal{H}$. For more information about topologies in $L(\mathcal{H})$ see [10].

Let us give some examples of linear operators.

Example 2.9 ([16])
Let $e_1, \ldots, e_n$ be an orthogonal basis in $\mathbb{C}^n$. Let A be any linear operator. Then we can define $a_{ij} = \langle Ae_j, e_i\rangle$, and the operator A can be identified with the matrix

$$A = [\langle Ae_j, e_i\rangle] = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}. \tag{2.1}$$

Example 2.10 ([16])
An interesting example of an operator in $\mathbb{C}^n$ is a *Jordan block operator* $\mathcal{J}_n \in L(\mathbb{C}^n)$, $\mathcal{J}_n(z_1, z_2, \ldots, z_n) = (0, z_1, z_2, \ldots, z_{n-1})$. It can be identified with the matrix

$$\mathcal{J}_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{2.2}$$

The number n is called the size of the Jordan block.

Example 2.11 ([23, Chap. 3])
Let $S \in L(l^2_+)$ be the *unilateral shift* defined as

$$S(h_0, h_1, \ldots) = (0, h_0, h_1, \ldots) \quad \text{for} \quad h = (h_0, h_1, \ldots) \in l^2_+. \tag{2.3}$$

If $e_0, e_1, \ldots$ is the standard orthogonal basis in l^2_+, (i.e. $e_k = (0, \ldots, 0, \overset{(k)}{1}, 0, \ldots)$, $k \in \mathbb{N}$), then S has a matrix representation

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \tag{2.4}$$

Example 2.12 ([28, Chap. 3],[27, Chap. 3])
When we consider the Hilbert space generated by a discrete stochastic process $\{\xi_n\}_{n\in\mathbb{N}}$, then there is a natural shift operator $T(\xi_n) = \xi_{n+1}$ for $n \in \mathbb{N}$. If we consider the Hilbert space generated by a continuous stochastic process $\{\xi_t\}_{t\geqslant 0}$, then we can define the operator T_s for $s \geqslant 0$ by $T_s(\xi_t) = \xi_{t+s}$, $t \geqslant 0$.

3. Invariant subspaces

Let as consider an operator T in a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} \geqslant 2$, and let as take a closed subspace $\mathcal{L} \subset \mathcal{H}$, (in what follows only closed subspaces will be considered). By $P_{\mathcal{L}}$ we denote the orthogonal projection onto the subspace $\mathcal{L}$. We will call the subspace $\mathcal{L}$ *invariant* for T if $T\mathcal{L} \subset \mathcal{L}$, i.e., for any $x \in \mathcal{L}$ the vector Tx also belongs to $\mathcal{L}$. It is a trivial observation that the whole space $\mathcal{H}$ and the zero space $\{0\}$ are invariant for any operator T. Thus we will be interested only in a nontrivial invariant subspace ($\mathcal{L} \neq \mathcal{H}$, $\mathcal{L} \neq \{0\}$).

Example 3.1
Let us consider (only in this example) the real Hilbert space $\mathbb{R}^2$, with the standard inner product, and an operator given by the matrix $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Since $\dim \mathbb{R}^2 = 2$, thus a nontrivial invariant subspace have to have a dimension one. So it has to be equal to $\mathbb{R}\, v$ for some vector $v \in \mathbb{R}$, $v = (v_1, v_2)$, $v \neq 0$. Thus $Tv = \lambda v$, which leads to equality $\lambda^2 + 1 = 0$. Since this equality has no solution in $\mathbb{R}$, we get to the contradiction. Hence, even in the space $\mathbb{R}^2$, there are operators with only trivial invariant subspaces.

The example above shows why we always assume that underlying Hilbert space is complex when we talk about invariant subspaces.

Remark 3.2
Let $T \in L(\mathcal{H})$ be any operator and let $\mathcal{L} \subset \mathcal{H}$ a be nontrivial closed subspace. Then $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}^{\perp}$, and the operator T can be represented metrically as $T = \left[\begin{array}{c:c} T_{11} & T_{12} \\ \hdashline T_{21} & T_{22} \end{array}\right]$. Note that $\mathcal{L}$ is invariant for T (i.e., $T\mathcal{L} \subset \mathcal{L}$) if and only if $T_{21} = 0$, i.e., T can be represented as $T = \left[\begin{array}{c:c} T_{11} & T_{12} \\ \hdashline 0 & T_{22} \end{array}\right]$. Hence a nontrivial invariant subspace allows us to simplify the representation of the operator.

Remark 3.3
Let $T \in L(\mathcal{H})$ be any operator and $\lambda \in \mathbb{C}$ be its eigenvalue. Then there is a non zero vector v such that $Tv = \lambda v$. Hence the subspace $\mathbb{C}\, v$ is invariant for T.

Now let us consider an operator $A \in L(\mathcal{H})$ in a finite dimensional Hilbert space $\mathcal{H}$ (which is in fact unitarily equivalent to $\mathbb{C}^n$). Then the existence of an eigenvalue λ with an eigenvector v (i.e., $Av = \lambda v$) is equivalent to $v \in \ker(A - \lambda I)$. Hence λ is an eigenvalue of A, if $A - \lambda I$ is not invertible. In other words, the characteristic polynomial of A, $z \to w_A(z) = \det(A - z\, I)$ equals to 0 at λ. But the fundamental theorem of algebra says that such λ always exists. Thus Remark 3.3 gives us the following.

Proposition 3.4
Let $A \in L(\mathcal{H})$, where $\mathcal{H}$ is a finite dimensional complex Hilbert space ($\dim \mathcal{H} \geqslant 2$). *Then A has a nontrivial invariant subspace.*

Unfortunately there are operators which does not have any eigenvalue.

Example 3.5
Let $S \in L(l^2_+)$ be the unilateral shift given by equation (2.3). Assume that λ is an eigenvalue of S with an eigenvector $h = (h_0, h_1, \dots) \neq 0$. Hence

$$Sh = (0, h_0, h_1, \dots) = (\lambda h_0, \lambda h_1, \dots) = \lambda(h_0, h_1, \dots).$$

If $\lambda \neq 0$, then we get $h = 0$. If $\lambda = 0$, then $\|h\| = \|Sh\| = 0$ implies also that $h = 0$. Hence we get a contradiction.

Now let $k_\lambda = (1, \lambda, \lambda^2, \dots)$ with $|\lambda| < 1$. Then $k_\lambda \in l^2_+$. The subspace $H_\lambda = \{k_\lambda\}^\perp$ is invariant for S. Indeed, if $h = (h_0, h_1, \dots) \in \mathcal{H}_\lambda$, then

$$\langle Sh, k_\lambda\rangle = \langle (0, h_0, h_1, \dots), (1, \lambda, \lambda^2, \dots)\rangle = \bar{\lambda}\langle h, k_\lambda\rangle = 0$$

and $Sh \in \mathcal{H}_\lambda$. Thus $\mathcal{H}_\lambda$ is invariant for S. Hence the unilateral shift S has a nontrivial invariant subspace but it has not any eigenvalue.

We have shown that each operator in finite dimensional Hilbert space and the unilateral shift have a nontrivial invariant subspace. There is a long list of operators in Hilbert spaces having a nontrivial invariant subspace: compact operators ([2]), operators which commute with compact operator ([19]), normal operators (consequence of spectral theorem, [9, Theorem IX 2.2]), subnormal operators ([7]), contractions with spectrum containing the unit circle ([8]), polynomially bounded operators with spectrum containing the unit circle ([1]), but still there is no solution of the general problem stated probably by John von Neumann:

Open problem 3.6
Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space. Let $T \in L(\mathcal{H})$ be any operator. Does T have a nontrivial invariant subspace, i.e., weather there exists $\mathcal{L} \subset \mathcal{H}$, $\mathcal{L} \neq \mathcal{H}$, $\mathcal{L} \neq \{0\}$ such that $T\mathcal{L} \subset \mathcal{L}$.

The problem seems to be even more interesting, since there are examples of Banach spaces and operators on those spaces, which do not have any nontrivial invariant subspace, see [14, 24, 25].

Investigating an operator, which has an invariant subspace, we obtain certain information about the operator and about its structure. Sometimes we can obtain a description of all its invariant subspaces and they, in some sense, describe the operator itself. Let $Lat\,T$ denote the set of all closed invariant subspaces for the operator T, i.e.,

$$Lat\,T = \{\mathcal{L} \subset \mathcal{H} : T\mathcal{L} \subset \mathcal{L}\}. \tag{3.1}$$

The set $Lat\,T$ from the algebraic point of view is a lattice with the intersection $\wedge$ of subspaces and spanning $\vee$ of subspaces as operations.

Example 3.7
Let $\mathcal{J}_n$ be the Jordan block operator of size n given by the matrix (2.2). Then it is easy to note that $Lat\,\mathcal{J}_n = \big\{\{0\}, \{0\} \oplus \mathbb{C}, \{0\} \oplus \mathbb{C}^2, \dots, \{0\} \oplus \mathbb{C}^{n-1}, \mathbb{C}^n\big\}$.

When we consider operators in a finite dimensional Hilbert space, we can investigate their characteristic polynomials. By the fundamental theorem of algebra we obtain all its eigenvalues with multiplicities $\lambda_1, \dots, \lambda_k$. The classical Jordan theorem says

Theorem 3.8 ([16, Theorem 3.1.11])
Let $A \in L(\mathcal{H})$, where $\mathcal{H}$ is a finite dimensional complex Hilbert space. Let $\lambda_1, \dots, \lambda_k$ be eigenvalues of A with multiplicities. Then

$$A \simeq \left[\begin{array}{c:c:c:c} \mathcal{J}_{n_1} - \lambda_1 I_{n_1} & 0 & \cdots & 0 \\ \hdashline 0 & \mathcal{J}_{n_2} - \lambda_2 I_{n_2} & \cdots & 0 \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline 0 & 0 & \cdots & \mathcal{J}_{n_k} - \lambda_k I_{n_k} \end{array}\right],$$

for certain sizes $n_1, \dots n_k$ ($\simeq$ means similarity).

Recall also that we say that an operator $A \in L(\mathcal{H})$ is *similar* to operator $B \in L(\mathcal{H})$ if there is an invertible operator $S \in L(\mathcal{H})$ such that $A = S^{-1}BS$.

The theorem above can give us a description of an invariant subspace by using the similarity and Example 3.7.

Now let us come back to the unilateral shift $S \in L(l^2_+)$ given by (2.3). As we have mentioned in Example 2.6, the space l^2_+ can be identified with the Hardy space H^2. Moreover the operator $S \in L(H^2)$ has the form $(Sf)(z) = z\,f(z)$ for $f \in H^2$.

It is impossible to describe all invariant subspaces of the unilateral shift S as the operator on l^2_+, but Beurling characterized all invariant subspaces of the unilateral shift S as the operator on the Hardy space H^2.

Theorem 3.9 ([6])
Let $S \in L(H^2)$ be the unilateral shift given by $(Sf)(z) = z\,f(z)$ for $f \in H^2$. Then $\mathcal{M} \neq \{0\}$ is invariant subspace of S ($\mathcal{M} \in Lat\,S$) if and only if there exists a holomorphic function $\varphi\colon \mathbb{D} \to \mathbb{C}$ such that $|\varphi(z)| = 1$ a.e. $z \in \partial\mathbb{D}$ and $\mathcal{M} = \varphi H^2$.

(The function φ is formally defined on $\mathbb{D}$, but by its values on the unit circle $\partial\mathbb{D}$ we understand the radial limits, see Example 2.6.)

We can give examples of such functions φ fulfilling the theorem above.

1. $p(z) = z^n$,
2. $B(z) = \prod\limits_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{z-a_n}{1+\overline{a_n}z}, \quad |a_n| \leqslant 1, \quad \sum\limits_{n=1}^{\infty} (1 - |a_n|) < \infty,$
3. $S(z) = \exp\left(-\frac{1+z}{1-z}\right)$.

The next interesting example of an operator, for which we can give a full description of its invariant subspaces, is the Voltera operator:

$$V\colon L^2[0,1] \to L^2[0,1], \quad (Vf)(x) = \int\limits_0^x f(t)\,dt \;\text{ for } f \in L^2[0,1].$$

For any $\alpha \in [0,1]$ let us denote by $\mathcal{M}_\alpha = \{f \in L^2[0,1] : f(t) = 0 \;\text{ for }\; t \in [0,\alpha]\}$. It is easy to see that $V\mathcal{M}_\alpha \subset \mathcal{M}_\alpha$, i.e., $\mathcal{M}_\alpha \in Lat\,V$. Moreover, all invariant subspaces for V are of this form.

Theorem 3.10 ([10, Theorem 28.3])
Let $V\colon L^2[0,1]\to L^2[0,1]$ be the Voltera operator given by $(Vf)(x)=\int_0^x f(t)\,dt$ for $f\in L^2[0,1]$. Then $Lat\,V=\{\mathcal{M}_\alpha\colon \alpha\in[0,1]\}$.

4. Reflexivity

When a given operator $T\in L(\mathcal{H})$ has a nontrivial invariant subspace, we can ask if it has enough invariant subspaces that they characterize the operator T itself. Recall that $Lat\,T$ denotes the set of all closed invariant subspaces for the operator T. Now let us consider the set of all operators which leave invariant all subspaces from $Lat\,T$, i.e.,

$$Alg\,Lat\,T=\{B\in L(\mathcal{H}) : B\mathcal{L}\subset\mathcal{L}\ \text{ for all }\ \mathcal{L}\in Lat\,T\}.$$

It is clear that all powers of T belong to $Alg\,Lat\,T$. It means that $T^n\in Alg\,Lat\,T$ for all $n\in\mathbb{N}$. Moreover, if p is a polynomial, $p(z)=a_0+a_1z+\cdots+a_nz^n$, then $p(T)=a_0I+a_1T+\cdots+a_nT^n\in Alg\,Lat\,T$. It is easy to observe that $Alg\,Lat\,T$ is, from the algebraic point of view, an algebra. From the topological point of view, $Alg\,Lat\,T$ is closed in the weak operator topology (WOT). Let $\mathcal{W}(T)$ denote the smallest algebra containing the operator T, the identity $I_{\mathcal{H}}$, and closed in the weak operator topology. In other words, it is a WOT closure of the set

$$\{p(T) : p\ \text{ is a polynomial}\}.$$

From the discussion above $\mathcal{W}(T)\subset Alg\,Lat\,T$. Following D. Sarason (see [26]) we will call the operator *reflexive* if $\mathcal{W}(T)=Alg\,Lat\,T$.

We mean that invariant subspaces characterize the operator T in the sense that $Alg\,Lat\,T$ is as small as it can be, i.e., $Alg\,Lat\,T=\mathcal{W}(T)$.

The notion of reflexivity is interesting even in a finite dimensional case.

Example 4.1
The operator $T_1=\begin{bmatrix}0&0\\1&0\end{bmatrix}\in L(\mathbb{C}^2)$ is not reflexive. Indeed,

$$\mathcal{W}(T_1)=\{\alpha I_{\mathbb{C}^2}+\beta T_1 : \alpha,\beta\in\mathbb{C}\}=\left\{\begin{bmatrix}\alpha&0\\\beta&\alpha\end{bmatrix} : \alpha,\beta\in\mathbb{C}\right\},$$

while

$$Lat\,T_1=\{\{0\}\oplus\{0\},\{0\}\oplus\mathbb{C},\mathbb{C}\oplus\mathbb{C}\}.$$

It is straightforward that subspace $\{0\}\oplus\mathbb{C}$ is invariant for the operator $\begin{bmatrix}\alpha&0\\\beta&\gamma\end{bmatrix}$ for any $\alpha,\beta,\gamma\in\mathbb{C}$. In fact

$$Alg\,Lat\,T_1=\left\{\begin{bmatrix}\alpha&0\\\beta&\gamma\end{bmatrix} : \alpha,\beta,\gamma\in\mathbb{C}\right\}\quad\text{and}\quad\mathcal{W}(T_1)\subsetneq Alg\,Lat\,T_1.$$

Example 4.2
The operator $T_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus [0] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in L(\mathbb{C}^3)$ is reflexive. Indeed, observe firstly that $\mathcal{W}(T_2) = \left\{ \begin{bmatrix} \alpha & 0 \\ \beta & \alpha \end{bmatrix} \oplus [\alpha] : \alpha, \beta \in \mathbb{C} \right\}$. It is easy to see that

$$\mathbb{C}^2 \oplus \{0\}, \{0\} \oplus \{0\} \oplus \mathbb{C}, \{0\} \oplus \mathbb{C} \oplus \{0\} \in Lat\, T_2.$$

Hence if $B = \begin{bmatrix} \alpha & \alpha_1 & \alpha_2 \\ \beta & \gamma & \alpha_3 \\ \alpha_4 & \alpha_5 & \delta \end{bmatrix} \in Alg\, Lat\, T_2$, the previous observation leads to $B = \begin{bmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \delta \end{bmatrix}$. Since subspaces $\{(x, y, x) : x, y \in \mathbb{C}\}$, $\{(0, x, x) : x \in \mathbb{C}\}$ belong to $Lat\, T_2$, we obtain $\alpha = \delta$ and $\gamma = \delta$. Hence $Alg\, Lat\, T_2 = W(T_2)$.

Example 4.3
Let $\mathcal{J}_n$ be the Jordan block operator of size n given by the matrix (2.2). If p is a polynomial, $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, then

$$p(\mathcal{J}_n) = \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & \cdots & a_0 & 0 \\ a_{n-1} & a_{n-2} & \cdots & \cdots & a_1 & a_0 \end{bmatrix}.$$

Hence

$$\mathcal{W}(\mathcal{J}_n) = \left\{ \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 & 0 \\ a_{n-1} & a_{n-2} & \cdots & \cdots & a_1 & a_0 \end{bmatrix} : a_i \in \mathbb{C} \right\}.$$

On the other hand, it is easy to see that

$$Lat\, \mathcal{J}_n = \{\{0\}, \{0\} \oplus \mathbb{C}, \{0\} \oplus \mathbb{C}^2, \ldots, \{0\} \oplus \mathbb{C}^{n-1}, \mathbb{C}^n\}.$$

Thus each element of $Alg\, Lat\, \mathcal{J}_n$ has to be lower triangular and

$$Alg\, Lat\, \mathcal{J}_n = \left\{ \begin{bmatrix} a_{00} & 0 & \cdots & \cdots & 0 \\ a_{10} & a_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n-1,0} & a_{n-1,1} & \cdots & \cdots & a_{n-1,n-1} \end{bmatrix} : a_{ij} \in \mathbb{C} \right\}.$$

Finally, $\mathcal{W}(\mathcal{J}_n) \subsetneq Alg\, Lat\, \mathcal{J}_n$ and $\mathcal{J}_n$ is not reflexive.

In fact the reflexive operators in a finite dimensional complex Hilbert space were completely characterized by Deddens and Fillmore [12]. For simplicity, we only present the result concerning a nilpotent operator. Recall that an operator $T \in L(\mathcal{H})$ is called *nilpotent* if and only if there is $n \in \mathbb{N}$ such that $T^n = 0$.

The main tool used in this characterization is the classical Jordan theorem, Theorem 3.8. If T is a nilpotent in a finite dimensional Hilbert space, Theorem 3.8 gives similarity:

$$T \simeq \left[\begin{array}{c:c:c:c} \mathcal{J}_{n_1} & 0 & \cdots & 0 \\ \hdashline 0 & \mathcal{J}_{n_2} & \cdots & 0 \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline 0 & 0 & \cdots & \mathcal{J}_{n_k} \end{array}\right] \tag{4.1}$$

for certain sizes $n_1, \dots n_k$. It is not hard to see that reflexivity is kept under similarity. That means that if A is similar to B, then A is reflexive if and only if B is reflexive.

Theorem 4.4 ([12])
Let $T \in L(\mathbb{C}^n)$ be a nilpotent. The operator T is reflexive if and only if the difference of size of the two largest blocks in the representation (4.1) *is at most* 1.

The following examples illustrate the theorem above.

Example 4.5
The operator $T \in L(\mathbb{C}^7)$, $T = \begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 0&1&0 \end{bmatrix} \oplus \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{bmatrix}$ is reflexive.

Example 4.6
The operator $T \in L(\mathbb{C}^7)$, $T = \begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 0&1&0 \end{bmatrix} \oplus \begin{bmatrix} 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix}$ is not reflexive.

Example 4.7
Let $T \in L(\mathbb{C}^{n_1})$, $n_1 = \frac{1}{2}(n+2)(n-1)$, $T = \mathcal{J}_2 \oplus \mathcal{J}_3 \oplus \cdots \oplus \mathcal{J}_n$, i.e.,

$$T = \begin{bmatrix} 0&0 \\ 1&0 \end{bmatrix} \oplus \begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 0&1&0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0&0&\cdots&0&0 \\ 1&0&\cdots&0&0 \\ 0&1&\cdots&0&0 \\ \vdots&\vdots&\ddots&\vdots&\vdots \\ 0&0&\cdots&1&0 \end{bmatrix}.$$

Then T is reflexive.

Example 4.8
The operator $T \in L(\mathbb{C}^{n(n+1)})$, $T = \mathcal{J}_2 \oplus \mathcal{J}_4 \oplus \cdots \oplus \mathcal{J}_{2n}$,

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

is not reflexive.

Talking about operators on infinite dimensional Hilbert spaces recall that Sarason proved reflexivity of the unilateral shift operator $S \in L(l^2_+)$.

Theorem 4.9 ([26])
Let $S \in L(l^2_+)$ be the unilateral shift. Then S is reflexive.

To prove reflexivity in Example 4.2 we have chosen very specific invariant subspaces. The idea of the proof of Theorem 4.9 is based on choosing subspace $\mathcal{H}_\lambda = \{k_\lambda\}^\perp$, where $k_\lambda = (1, \lambda, \lambda^2, \ldots)$, $|\lambda| < 1$. An easy observation is that $S\mathcal{H}_\lambda \subset \mathcal{H}_\lambda$. Next, using this subspace and the properties of the unilateral shift we obtain the reflexivity.

The natural representation of the unilateral shift is

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Thus S can be seen as the Jordan block of "infinite" size and by comparing Example 4.3 and Theorem 4.9 we conclude that S behaves „better " than the Jordan block $\mathcal{J}_n$ of size n.

Now we can ask about the operators in Example 4.7 and 4.8, if we change a finite orthogonal sum to infinite one. Would they be reflexive? The following theorem holds

Theorem 4.10 ([5])
The operator $S_1 = \mathcal{J}_2 \oplus \mathcal{J}_3 \oplus \cdots \oplus \mathcal{J}_k \oplus \ldots$, i.e.,

$$S_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \oplus \ldots$$

is reflexive.

Remark 4.11
The main theorem in [5] shows, in fact, that operator S_2 is also reflexive if $S_2 = \mathcal{J}_2 \oplus \mathcal{J}_4 \oplus \cdots \oplus \mathcal{J}_{2k} \oplus \ldots$, i.e.,

$$S_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \oplus \ldots .$$

Note that S_1 and S_2 have the following matrix representations

$$S_1 = \left[\begin{array}{cc:ccc:cccc:ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ \hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right] \quad S_2 = \left[\begin{array}{cc:cccc:cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\ \hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right].$$

In the equation (3.1) we have defined the lattice of invariant subspaces of a single operator. Now, for a given algebra of operators $\mathcal{W} \subset L(\mathcal{H})$, we define

$$Lat\,\mathcal{W} = \{\mathcal{L} \subset \mathcal{H} : A\mathcal{L} \subset \mathcal{L} \ \text{ for } \ A \in \mathcal{W}\ \} = \bigcap_{A \in \mathcal{W}} Lat\,A.$$

We have noticed above that, for a given operator $T \in L(\mathcal{H})$, if the subspace $\mathcal{L}$ is invariant for T, $\mathcal{L} \in Lat\,T$, thus it is also invariant for any operator $A \in \mathcal{W}(T)$. Hence $Lat\,T = Lat\,\mathcal{W}(T)$.

Now let $\mathcal{W} \subset L(\mathcal{H})$ and $\mathcal{W}$ be an algebra. Then

$$Alg\,Lat\,\mathcal{W} = \{B \in L(\mathcal{H}) : B\mathcal{L} \subset \mathcal{L} \ \text{ for all } \ \mathcal{L} \in Lat\,\mathcal{W}\}.$$

The algebra $\mathcal{W}$ is called *reflexive* if $\mathcal{W} = Alg\,Lat\,\mathcal{W}$. It is straightforward that the definition of reflexivity of an operator T coincides with the definition of reflexivity of the algebra $\mathcal{W}(T)$, i.e., T is reflexive if and only if $\mathcal{W}(T)$ is reflexive. As an example of a reflexive algebra, which is not singly generated, can be taken the algebra of lower triangular matrices.

Theorem 4.12
The algebra $\mathcal{A}_n \subset L(\mathbb{C}^n)$ is reflexive, where

$$\mathcal{A}_n = \left\{ \begin{bmatrix} a_{00} & 0 & \cdots & \cdots & 0 \\ a_{10} & a_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n-1,0} & a_{n-1,1} & \cdots & \cdots & a_{n-1,n-1} \end{bmatrix} : a_{ij} \in \mathbb{C} \right\}.$$

It is easy to see that $Lat\,\mathcal{A}_n = \{\{0\}, \{0\} \oplus \mathbb{C}, \{0\} \oplus \mathbb{C}^2, \dots, \{0\} \oplus \mathbb{C}^{n-1}, \mathbb{C}^n\}$ so

$$Alg\,Lat\,A_n = \left\{ \begin{bmatrix} a_{00} & 0 & \cdots & \cdots & 0 \\ a_{10} & a_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n-1,0} & a_{n-1,1} & \cdots & \cdots & a_{n-1,n-1} \end{bmatrix} : a_{ij} \in \mathbb{C} \right\}.$$

An investigation of reflexivity leads to the natural extension of the definition to subspaces of operators. To understand the idea let us consider an algebra $\mathcal{W} \subset L(\mathcal{H})$ and its lattice of invariant subspaces $Lat\,\mathcal{W}$. Observe that $\mathcal{L} \in Lat\,\mathcal{W}$ is equivalent to the property that for any $x \in \mathcal{L}$, $y \perp \mathcal{L}$ and $A \in \mathcal{W}$ we have $\langle Ax, y\rangle = 0$. Moreover, saying that $B \in Alg\,Lat\,\mathcal{W}$ is equivalent to the condition that for any $\mathcal{L} \in Lat\,\mathcal{W}$ and $x \in \mathcal{L}$, $y \perp \mathcal{L}$ we have $\langle Bx, y\rangle = 0$.

Following [18] for a subspace $\mathcal{M} \subset L(\mathcal{H})$ we can define

$$Ref\,\mathcal{M} = \{B \in L(\mathcal{H}) : \forall_{x,y\in\mathcal{H}}\ (\langle Ax, y\rangle = 0 \ \text{ for all } \ A \in \mathcal{M}) \Longrightarrow \langle Bx, y\rangle = 0\}.$$

The subspace $\mathcal{M}$ is called *reflexive* if and only if $\mathcal{M} = Ref\,\mathcal{M}$. Clearly from above if $\mathcal{M}$ is an algebra, both definitions coincide.

Let us give some examples of reflexive and not reflexive subspaces of operators.

Example 4.13 ([4])
Let $T \in L(\mathcal{H})$. Then a subspace $\mathbb{C}\,T$ is reflexive. In other words a one dimensional subspace is always reflexive.

Example 4.14 ([4])
Let $\mathcal{T}_n \subset L(\mathbb{C}^n)$ be the space of all Toeplitz matrices, i.e.,

$$\mathcal{T}_n = \left\{ \begin{bmatrix} a_0 & a_{-1} & \cdots & \cdots & a_{-n} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-n+1} \\ a_2 & a_1 & a_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_n & a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix} : a_{-n}, \dots, a_{-1}, a_0, a_1, \dots, a_n \in \mathbb{C} \right\}.$$

Note that $\mathcal{T}_n$ is not an algebra and it is clear that it is not reflexive.

5. Hyperreflexivity

Let $\mathcal{W}$ be an algebra closed in the norm topology, and let $B \in L(\mathcal{H})$ be any operator. Then we can define the „usual" distance in the standard way:

$$\operatorname{dist}(B, \mathcal{W}) = \inf\{\|B - A\| : A \in \mathcal{W}\} = \inf\{\sup\{\|Bx - Ax\| : \|x\| \leqslant 1\} : A \in \mathcal{W}\}.$$

We can define the distance from the operator B to $\mathcal{W}$ „taking into account in B only this part, which is not invariant for elements of the lattice $Lat\,\mathcal{W}$". Precisely,

$$\begin{aligned}\alpha(B,\mathcal{W}) &= \sup\{\inf\{\|P_{\mathcal{L}}^{\perp}(B-A)P_{\mathcal{L}}\| : A\in\mathcal{W}\} : \mathcal{L}\in Lat\,\mathcal{W}\}\\ &= \sup\{\inf\{\|P_{\mathcal{L}}^{\perp}BP_{\mathcal{L}}\| : A\in\mathcal{W}\} : \mathcal{L}\in Lat\,\mathcal{W}\}\\ &= \sup\{\|P_{\mathcal{L}}^{\perp}BP_{\mathcal{L}}\| : \mathcal{L}\in Lat\,\mathcal{W}\}.\end{aligned} \tag{5.1}$$

As above, instead of considering the subspaces $\mathcal{L}$ in $Lat\,\mathcal{W}$ we can use suitable vectors $x\in\mathcal{L}$, $y\in\mathcal{L}^{\perp}$ and following the same idea we can extend the definition of the distance α to subspaces of operators. Namely, let $\mathcal{M}\subset L(\mathcal{H})$ be a subspace and let $B\in L(\mathcal{H})$. Then

$$\begin{aligned}\alpha(B,\mathcal{M}) &= \sup\{|\langle Bx,y\rangle| : \|x\|,\|y\|\leqslant 1, \langle Ax,y\rangle = 0,\ A\in\mathcal{M}\}\\ &= \sup\{|\langle Bx,y\rangle - \langle Ax,y\rangle| : \|x\|,\|y\|\leqslant 1, \langle Ax,y\rangle = 0,\ A\in\mathcal{M}\}.\end{aligned}$$

If $\mathcal{M}$ is an algebra, then this definition coincides with (5.1). It is easy to see that always $\alpha(B,\mathcal{M})\leqslant \operatorname{dist}(B,\mathcal{M})$. We can ask, whether we can control the usual distance dist by α distance. Now, following [3] and [18], a norm closed subspace $\mathcal{M}\subset L(\mathcal{H})$ will be called *hyperreflexive* if there is $c>0$ such that

$$\operatorname{dist}(B,\mathcal{M})\leqslant c\,\alpha(B,\mathcal{M}) \quad \text{for all} \quad B\in L(\mathcal{H}). \tag{5.2}$$

The smallest constant fulfilling (5.2) will be denoted by $\kappa_{\mathcal{M}}$.

To see that hyperreflexivity is a stronger property than reflexivity, note that $\langle Bx,y\rangle = 0$ for all x,y such that $\langle Ax,y\rangle = 0$ for all $A\in\mathcal{M}$ if and only if $B\in Ref\,\mathcal{W}$. Hence if $B\in Ref\,\mathcal{W}$, then the right hand side of (5.2) equals 0, thus $\operatorname{dist}(B,\mathcal{M})=0$, It means that $B\in\mathcal{M}$, since $\mathcal{M}$ is norm closed.

The following lemma about the quotient space $L(\mathcal{H})/\mathcal{M}$ gives a deeper understanding of the notion of hyperreflexivity.

Lemma 5.1
Let $\mathcal{M}\subset L(\mathcal{H})$ be a norm closed subspace. Then:

1. dist: $L(\mathcal{H})/\mathcal{M}\to\mathbb{R}_+$ *is a norm in* $L(\mathcal{H})/\mathcal{M}$*,*
2. α: $L(\mathcal{H})/\mathcal{M}\to\mathbb{R}_+$ *is a seminorm in* $L(\mathcal{H})/\mathcal{M}$*,*
3. *If $\mathcal{M}$ is reflexive, then α is a norm in $L(\mathcal{H})/\mathcal{M}$.*

To prove 5.1, recall from the above that if $\alpha(B,\mathcal{M})=0$ for some $B\in L(\mathcal{H})$, then $B\in Ref\,\mathcal{M}$. By reflexivity of $\mathcal{M}$, we get $B\in\mathcal{M}$.

In a view of Lemma 5.1 the question of hyperreflexivity of a subspace $\mathcal{M}$ is equivalent to the question of the equivalence of the norms α and dist in the space $L(\mathcal{H})/\mathcal{M}$.

Note that if $\dim\mathcal{H}<\infty$ then also $\dim L(\mathcal{H})<\infty$ and $\dim L(\mathcal{H})/\mathcal{M}<\infty$ for any $\mathcal{M}\subset L(\mathcal{H})$. Since all the norms in a finite dimensional subspace are equivalent, we have the following

Proposition 5.2
Let $\mathcal{M} \subset L(\mathcal{H})$ *and* $\dim \mathcal{H} < \infty$. *Then* $\mathcal{M}$ *is reflexive if and only if and only if* $\mathcal{M}$ *is hyperreflexive.*

Theorem 4.12 says that the algebra of all lower triangular matrices $\mathcal{A}_n \subset L(\mathbb{C}^n)$ is reflexive. Thus by Proposition 5.2 it is also hyperreflexive.

In fact, following Arverson we have

Theorem 5.3 ([3])
Let $\mathcal{A}_n \subset L(\mathbb{C}^n)$ *be the algebra of all lower triangular matrices. Then* $\mathcal{A}_n$ *is hyperreflexive and* $\kappa_{\mathcal{A}_n} = 1$.

By Theorem 4.4 the operator $\mathcal{J}_n \oplus \mathcal{J}_{n-1}$ (the orthogonal sum of the Jordan blocks of size n and of size $n-1$) is reflexive, i.e., $\mathcal{W}(\mathcal{J}_n \oplus \mathcal{J}_{n-1})$ is reflexive. Hence it is hyperrreflexive. It is natural to ask about the constant $\kappa_{\mathcal{W}(\mathcal{J}_n \oplus \mathcal{J}_{n-1})}$.

Open problem 5.4
Is $\kappa_{\mathcal{W}(\mathcal{J}_n \oplus \mathcal{J}_{n-1})}$ bounded by a constant independent on n?

The next natural question put forward by Kraus and Larson [18] is whether the reflexivity and the hyperreflexivity of a subspace are equivalent, when the underlying Hilbert space is infinite dimensional but the dimension of $\mathcal{M}$ is finite, $\dim \mathcal{M} < \infty$. The following is true

Theorem 5.5 ([21])
Let $\mathcal{M} \subset L(\mathcal{H})$ *and* $\dim \mathcal{H} = \infty$, $\dim \mathcal{M} < \infty$. *Then* $\mathcal{M}$ *is reflexive if and only if* $\mathcal{M}$ *is hyperreflexive.*

One of the tools in the proof of the above is classical Helly's Theorem (1923).

Theorem 5.6 ([15])
Let $\{X_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ *be a sequence of nonempty closed convex sets. Then*

$$\forall_{\{n_k\}_{k=1}^{d+1}} \quad \bigcap_{k=1}^{d+1} X_{n_k} \neq \emptyset \Longrightarrow \bigcap_{n=1}^{\infty} X_n \neq \emptyset.$$

Another interesting question concerning the theorem above is whether the $\kappa_{\mathcal{M}}$ depends on the dimension of $\mathcal{M}$. The example bellow shows that even for a subspace of dimension 2 in a three dimensional Hilbert space the constant $\kappa_{\mathcal{M}}$ can be arbitrary large.

Example 5.7
Let $\varepsilon > 0$ and $A_{1,\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus [\varepsilon]$, $A_{2,\varepsilon} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus [0]$. Let $\mathcal{M}_\varepsilon = span\,\{A_{1\,\varepsilon}, A_{2\,\varepsilon}\}$. Then $\dim \mathcal{M}_\varepsilon = 2$. It can be shown, that $\mathcal{M}_\varepsilon$ is reflexive, and hence hyperreflexive, but $\kappa_{\mathcal{M}_\varepsilon} > \frac{2}{\varepsilon}$.

This gives a possibility to construct an example of a reflexive, but not hyperreflexive subspace.

Example 5.8 ([21])
Let $\mathcal{M}_\varepsilon$ be as in Example 5.7. Consider $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_{\frac{1}{2}} \oplus \mathcal{M}_{\frac{1}{3}} \oplus \cdots$. Then it can be shown that $\mathcal{M}$ is reflexive, but not hyperreflexive.

Davidson showed hyperreflexivity of the unilateral shift S.

Theorem 5.9 ([11])
Let $S \in L(l^2_+)$ be the unilateral shift. Then $\mathcal{W}(S)$ is hyperreflexive and $\kappa_{\mathcal{W}(S)} < 18$. (It was shown in [17] that $\kappa_{\mathcal{W}(S)} < 13$.)

Now we can ask about hyperreflexivity of operators appearing in Theorem 4.13.

Theorem 5.10 ([22])
Let

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \oplus \ldots .$$

Then T is hyperreflexive and $\kappa_{\mathcal{W}(T)} < 11$.

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