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Line arrangements and Harbourne constants

Streszczenie. Liniowe stałe Harbourne'a zostały wprowadzone w [1] w związku z Bounded Negativity Conjecture. Stałe te są powiązane z konfiguracjami prostych. Konfiguracje można rozpatrywać nad dowolnym ciałem liczbowym. Pewne konfiguracje da się zrealizować geometrycznie tylko nad pewnymi ciałami. W pracy tej zajmujemy się stałymi Harbourne'a: rzeczywistymi, zespolonymi oraz absolutnymi. Głównym wynikiem zaprezentowanym w tej pracy jest wskazanie najmniejszej liczby d_b takiej, że:

$$\mathcal{H}_L(\mathbb{C}, d_b) < \mathcal{H}_L(\mathbb{R}, d_b).$$

Przy okazji prowadzonych badań uzyskaliśmy i sformułowaliśmy warunek konieczny i wystarczający istnienia konfiguracji do 7 prostych rzeczywistych.

Abstract. Linear Harbourne constants have been introduced in [1] in connection with the Bounded Negativity Conjecture. These are numerical invariants associated to arrangements of lines. This notion depends on the configuration of lines: the number of lines and the number and multiplicity of points in which the lines intersect. It is well-known that certain arrangements can be realized only over certain fields. Thus Harbourne constants also depend on the ground field. In this note we consider three variants of Harbourne constants: complex, real and absolute (i.e. associated to arrangements defined over *some* field). Let $\mathcal{H}_L(\mathbb{K}, d)$ be the minimum of all linear Harbourne constants of d lines defined over a field $\mathbb{K}$ (see Definition 1.4). In the present note, we will find the least number d_b such that

$$\mathcal{H}_L(\mathbb{C}, d_b) < \mathcal{H}_L(\mathbb{R}, d_b).$$

As a by-product, we establish necessary and sufficient conditions for the existence of arrangements of up to seven real lines.

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Słowa kluczowe: Harbourne constants, arrangements of lines, combinatorial arrangements.

[2] *Konfiguracje prostych i stałe Harbourne'a*

1. Preliminaries

In the present paper we study arrangements of lines in projective planes. We consider planes $\mathbb{P}^2(\mathbb{K})$ defined over an arbitrary field $\mathbb{K}$. An arrangement $\mathfrak{L}$ is a finite set of lines ℓ_i, $i = 1, \ldots, d$ in $\mathbb{P}^2(\mathbb{K})$.

Definition 1.1
A point $P \in \mathbb{P}^2(\mathbb{K})$ is a *singular point of an arrangement* $\mathfrak{L}$ if it is the intersection point of at least two lines from $\mathfrak{L}$. The number of lines from $\mathfrak{L}$ passing through a singular point P is called its *multiplicity* and we denote it by $\operatorname{mult}_{\mathfrak{L}}(P)$. The set of all singular points of an arrangement is denoted by $\wp(\mathfrak{L})$.

Definition 1.2
An arrangement $\mathfrak{L}$ of d lines is a *pencil* of lines if there exists a point $P \in \wp(\mathfrak{L})$ with $\operatorname{mult}_{\mathfrak{L}}(P) = d$. An arrangement $\mathfrak{L}$ of d lines is a *quasi-pencil* of lines if there exists a point $P \in \wp(\mathfrak{L})$ with $\operatorname{mult}_{\mathfrak{L}}(P) = d - 1$.

Definition 1.3 (Linear Harbourne constant of $\mathfrak{L}$ at $\wp(\mathfrak{L})$)
Let $\mathfrak{L} = \{\ell_\iota\}_{\iota=1}^{d}$ be an arrangement of d $\mathbb{K}$-lines in the projective plane, let $\wp(\mathfrak{L}) = \{P_\iota\}_{\iota=1}^{s}$ be the set of singular points of the arrangement. The linear Harbourne constant of $\mathfrak{L}$ at $\wp(\mathfrak{L})$ is:

$$\mathcal{H}_L(\mathbb{K}, \mathfrak{L}) := \frac{d^2 - \sum_{k=1}^{s} \operatorname{mult}_{\mathfrak{L}}(P_k)^2}{s}.$$

Definition 1.4 (Linear Harbourne constant of arrangements of d $\mathbb{K}$-lines)
The linear Harbourne constant of arrangements of d $\mathbb{K}$-lines is:

$$\mathcal{H}_L(\mathbb{K}, d) := \min_{\mathfrak{L}\colon\ \#\mathfrak{L}=d} \mathcal{H}_L(\mathbb{K}, \mathfrak{L}).$$

Definition 1.5 (Absolute linear Harbourne constant)
Let d be a given natural number, $d \geqslant 2$. The absolute linear Harbourne constant is:

$$\mathcal{H}_L(d) := \min_{\mathbb{K}} \mathcal{H}_L(\mathbb{K}, d).$$

2. Open problems

There is a number of open problems which we would like to address here.

Problem 2.1
Determine for which solutions of the diophantine combinatorial equation in Definition 3.1, there exists a line arrangement with this data.

Problem 2.2
Does there exist a universal necessary and sufficient condition for the existence of a line arrangement? Only some necessary criteria are known.

Problem 2.3
Determine Harbourne constants of all line arrangements.

Problem 2.4
No general method of computing Harbourne constants is known.

After 2015, there appeared some partial answers to these questions. Partial answers to Problems 2.1 and 2.2 are given in this paper. Partial answers to Problem 2.3 are:

- [7]: for the values of $\mathcal{H}_L(d)$ and $\mathcal{H}_L(\mathbb{C}, d)$ if $d \leqslant 10$;
- [6]: for the values of $\mathcal{H}_L(\mathbb{R}, \mathfrak{L})$ and $\mathcal{H}_L(\mathbb{R}, d)$ if $d \leqslant 7$;
- [2]: for the values of $\mathcal{H}_L(d)$ if $d \leqslant 31$;
- this paper: for the values of $\mathcal{H}_L(\mathbb{R}, \mathfrak{L})$ and $\mathcal{H}_L(\mathbb{R}, d)$ if $d \leqslant 9$.

Some observations and estimates related to Problem 2.4 appeared in [6], [7] and this paper, a number of interesting inequalities, estimates and theorems appeared also in [2].

3. Criteria for the existence of an arrangement of $\mathbb{K}$-lines

Definition 3.1 (Diophantine combinatorial equation)
Let $\mathfrak{L} = \{\ell_\iota\}_{\iota=1}^{d}$ be an arrangement of d $\mathbb{K}$-lines, let $\wp(\mathfrak{L}) = \{P_\iota\}_{\iota=1}^{s}$ be the set of singular points of $\mathfrak{L}$. Let $t_k := \#\{P_\iota : \mathrm{mult}_{\mathfrak{L}}(P_\iota) = k\}$. Then the following combinatorial equality holds:

$$\binom{d}{2} = \sum_{k=2}^{\infty} t_k \cdot \binom{k}{2}.$$

The equality above can be also treated as a *diophantine equation* in which t_k's are variables.

This equation with fixed d has a finite number of solutions. We know that every existing arrangement is given by exactly one solution, but not every solution to this equation corresponds to a geometrical arrangement (see Lemma 3.3). For this reason, we would like to find a set of criteria which will enable us to check whether certain solutions correspond to geometrical arrangements, and which do not. Indicating a necessary and sufficient condition for the exitence of the arrangements matching a given solution (for any number of lines) is still a problem challenging for mathematicians. Below we present a set of two criteria (C1 and C3) which is a necessary and sufficient condition for the existence of an arrangement up to 7 $\mathbb{R}$-lines. We tried to create a necessary and sufficient condition for the existence of the arrangement up to 9 $\mathbb{R}$-lines (we formulated a conjecture that this set will comprise Criterion C1, Criterion C2 and Criterion C3, later also Criterion C4), but it turned out this conjecture is false. This means that for $d \geqslant 8$ the problem is still open.

Observation 3.2 (Combinatorial quotient)
Let $T=(t_\iota)_{\iota=2}^{\infty}$ be a solution of the diophantine combinatorial equation. Then let

$$q(T) := \frac{d^2 - \sum_{k=2}^{d} k^2 t_k}{\sum_{k=2}^{d} t_k}.$$

It is easy to see that $\mathcal{H}_L(\mathbb{K},\mathfrak{L}) = q(T)$, when T is a solution to the combinatorial equation which corresponds to $\mathfrak{L}$.

Criterion C1 (Triangular Inequality)
Let $\mathfrak{L}=\{\ell_\iota\}_{\iota=1}^{d}$ be an arrangement of d $\mathbb{K}$-lines, $\wp(\mathfrak{L}) = \{P_\iota\}_{\iota=1}^{s}$ a set of singular points of the arrangement. Denote by $\mathrm{mult}_{\mathfrak{L}}(P_{a_1}), \mathrm{mult}_{\mathfrak{L}}(P_{a_2}), \mathrm{mult}_{\mathfrak{L}}(P_{a_3})$ the three largest consecutive multiplicities of points in $\wp(\mathfrak{L})$, then if $s \geqslant 3$, one has:

$$\sum_{i\in\{a,b,c\}} \mathrm{mult}_{\mathfrak{L}}(P_i) \leqslant d+3.$$

Criterion C2 (Quadrangle Inequality)
Let $\mathfrak{L}=\{\ell_\iota\}_{\iota=1}^{d}$ be an arrangement of d $\mathbb{K}$-lines, $\wp(\mathfrak{L}) = \{P_\iota\}_{\iota=1}^{s}$ the set of singular points of the arrangement. Denote by $\mathrm{mult}_{\mathfrak{L}}(P_{a_1}), \mathrm{mult}_{\mathfrak{L}}(P_{a_2}), \mathrm{mult}_{\mathfrak{L}}(P_{a_3})$, $\mathrm{mult}_{\mathfrak{L}}(P_{a_4})$ be four largest consecutive multiplicities of points in $\wp(\mathfrak{L})$, then if $s \geqslant 4$, one has:

$$\sum_{i\in\{a_1,a_2,a_3,a_4\}} \mathrm{mult}_{\mathfrak{L}}(P_i) \leqslant d+6.$$

Easy proofs of Criterium C1 and Criterium C2 can be found in [7]. None of these criteria is stronger than the other - it is illustrated in Paragraph 4. (for example: compare ($d=8$, T_{42}) and ($d=9$, T_{100})).

Criterion C3 (Melchior's inequality [4])

Let $\mathfrak{L}=\{\ell_\iota\}_{\iota=1}^{d}$ be an arrangement of d $\mathbb{R}$-lines. If the arrangement $\mathfrak{L}$ is not a pencil of lines, there is the inequality:

$$\sum_{k\geqslant 2} (k-3)t_k \leqslant -3 \ ,$$

which is of course equivalent to $\sum_{k\geqslant 4}(k-3)t_k + 3 \leqslant t_2$.
We present a proof of this inequality by showing its generalization – Theorem 3.4. Melchior's inequality is stronger than the Sylvester-Gallai theorem which says that $t_2 > 0$ (if the arrangement is not a pencil of lines) – for more details see [6].

Lemma 3.3
For every $d \geqslant 4$ there exists a solution of the diophantine combinatorial equation (Definition 3.1), that there does not exist a line arrangement with this data.

Proof. For $d = 4$ there is a known solution T: $t_3 = 2, t_k = 0$, for all $k \neq 3$. This solution cannot be realized over any field. For $d \geqslant 3$ we have an obvious equality $\binom{d-1}{2} + d = \binom{d}{2}$. Therefore, for every $d \geqslant 5$ there exists a solution:

$$T_*^d := \begin{cases} t_2 = d - 3 \\ t_3 = 1 \\ t_{d-1} = 1 \\ t_i = 0, \text{ for } i \notin \{2, 3, d-1\}. \end{cases}$$

For every $d \geqslant 5$, these solutions are excluded by Melchior's inequality. ■

3.1. Melchior and Hirzebruch type criteria

There is a natural generalization of Melchior's inequality. It seems to us that this result didn't show up in the literature, so we present our proof below.

THEOREM 3.4 (Generalization of Melchior's inequality)
Let $\mathfrak{L}$ be an arrangement of at least 3 lines in $\mathbb{P}^2(\mathbb{R})$ such that it is not a pencil. Then

$$\sum_{r>2n+1} (r - (2n+1))t_r + 2n + 1 \leq \sum_{r=2}^{2n} (2n + 1 - r)t_r.$$

Proof. Let us denote by p_r the number of regions in $\mathbb{P}^2(\mathbb{R})$ bounded by r segments (r-gons). Moreover,

- $f_0 := \sum_{r \geq 2} t_r$,
- $f_1 := \sum_{r \geq 2} r t_r = \frac{1}{2} \sum_{r \geq 2} r p_r$,
- $f_2 := \sum_{r \geq 2} p_r$.

Note that the equality for f_1 follows from counting segments once as edges of r-gons and once as segments containing two vertices. The Euler–Poincaré characteristic of $\mathbb{P}^2(\mathbb{R})$ gives us that

$$f_0 - f_1 + f_2 = 1. \tag{3.1}$$

We have

$$-(2n+1)f_0 + 2nf_1 + f_1 - (2n+1)f_2 = -(2n+1)$$

by multiplying equality (1) by $-(2n+1)$ with $n \geq 1$. Then

$$2nf_1 - (2n+1)f_2 = -(2n+1) - f_1 + (2n+1)f_0.$$

The left-hand side of the equation gives us

$$2nf_1 - (2n+1)f_2 = \sum_{r \geq 2} ((nr - 2) - 1)p_r.$$

We know that $p_2 = 0$ (sine $\mathfrak{L}$ is not a pencil) and $n \geq 1$, hence

$$\sum_{r \geq 2} ((nr - 2) - 1)p_r \geqslant 0.$$

Due to this reason

$$0 \leq -(2n+1) - f_1 + (2n+1)f_0.$$

Therefore

$$\sum_{r>2n+1} (r-(2n+1))t_r + 2n+1 \leq \sum_{r=2}^{2n} (2n+1-r)t_r.$$

Now for $n = 1$ we recover Melchior's inequality (Criterion C3). ∎

Lemma 3.5 (Hirzebruch's inequality [3, Theorem p.148])
Let $\mathfrak{L} = \{\ell_\iota\}_{\iota=1}^{d}$ be an arrangement of d $\mathbb{C}$-lines. If the arrangement $\mathfrak{L}$ is not a pencil or a quasi-pencil of lines, there is the inequality:

$$t_2 + \frac{3}{4}t_3 \geqslant d + \sum_{k=5}^{\infty}(k-4)t_k.$$

Below we present the strongest known inequality for line arrangements with $t_r = 0$ for $r > \frac{2}{3}k$.

Criterion C4 (Inequality stronger than Hirzebruch's inequality [5, Rem. 2.5])

Let $\mathfrak{L}$ be an arrangement of d $\mathbb{C}$-lines such that $t_r = 0$ for $r > \frac{2}{3}d$ Then

$$t_2 + \frac{3}{4}t_3 \geqslant d + \sum_{r=5}^{\infty}\left(\frac{r^2}{4} - r\right)t_r.$$

This is the strongest inequality of this kind. For more details consult [5].

4. Main results

Below, we present the tables which:

- prove a necessary and sufficient condition for the existence of arrangements of up to 7 $\mathbb{R}$-lines);
- enumerate values of all existing arrangements $\mathfrak{L}$ of d $\mathbb{R}$-lines ($\mathcal{H}_L(\mathbb{R}, \mathfrak{L})$), for d up to 9 and give values of $\mathcal{H}_L(\mathbb{R}, d)$;
- prove that $\exists_{d\geq 3}$ $\mathcal{H}_L(\mathbb{R}, d) \gneqq \mathcal{H}_L(d)$;
- prove that $\exists_{d\geq 3}$ $\mathcal{H}_L(\mathbb{R}, d) \gneqq \mathcal{H}_L(\mathbb{C}, d)$.

The existence of all arrangements of $\mathbb{R}$-lines, for which the values of Harbourne constants have been computed, has been checked by us empirically by means of drawings (we do not publish them because of their number and size). The non-existence of arrangements (n.e. - *not exists*) for solutions which do not give any arrangement in $\mathbb{P}^2(\mathbb{R})$ has been proven by either directly by criteria included in the previous sections or is reduced to these criteria by simple manipulations, e.g. adding or deleting a line.

4.1. Two $\mathbb{R}$-lines

	t_2	$C_{1(\leq 5)}$	$C_{2(\leq 8)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_1	1	$+_t$	$+_t$	$+_t$	$\mathbf{0}$

4.2. Three $\mathbb{R}$-lines

	t_2	t_3	$C_{1(\leq 6)}$	$C_{2(\leq 9)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_1	0	1	$+_t$	$+_t$	$+_t$	0
T_2	3	0	$+_{(6)}$	$+_t$	$+_{(-3)}$	$\mathbf{-1}$

4.3. Four $\mathbb{R}$-lines

	t_2	t_3	t_4	$C_{1(\leq 7)}$	$C_{2(\leq 10)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_1	0	0	1	$+_t$	$+_t$	$+_t$	0
T_2	0	2	0	$+_t$	$+_t$	$-_{(0)}$	n.e.
T_3	3	1	0	$+_{(7)}$	$+_{(9)}$	$+_{(-3)}$	$-1\frac{1}{4}$
T_4	6	0	0	$+_{(6)}$	$+_{(8)}$	$+_{(-6)}$	$\mathbf{-1\frac{1}{3}}$

4.4. Five $\mathbb{R}$-lines

	t_2	t_3	t_4	t_5	$C_{1(\leq 8)}$	$C_{2(\leq 11)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_1	0	0	0	1	$+_t$	$+_t$	$+_t$	0
T_2	1	1	1	0	$-_{(9)}$	$+_t$	$-_{(0)}$	n.e.
T_3	4	0	1	0	$+_{(8)}$	$+_{(10)}$	$+_{(-3)}$	$-1,4$
T_4	1	3	0	0	$-_{(9)}$	$+_{(11)}$	$-_{(-1)}$	n.e.
T_5	4	2	0	0	$+_{(8)}$	$+_{(10)}$	$+_{(-4)}$	$\mathbf{-1,5}$
T_6	7	1	0	0	$+_{(7)}$	$+_{(9)}$	$+_{(-7)}$	$\mathbf{-1,5}$
T_7	10	0	0	0	$+_{(6)}$	$+_{(8)}$	$+_{(-10)}$	$\mathbf{-1,5}$

4.5. Six $\mathbb{R}$-lines

	t_2	t_3	t_4	t_5	t_6	$C_{1(\leq 9)}$	$C_{2(\leq 12)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_1	0	0	0	0	1	$+_t$	$+_t$	$+_t$	0
T_2	2	1	0	1	0	$-_{(10)}$	$+_{(12)}$	$-_{(0)}$	n.e.
T_3	5	0	0	1	0	$+_{(9)}$	$+_{(11)}$	$+_{(-3)}$	$-1,5$
T_4	0	1	2	0	0	$-_{(11)}$	$+_t$	$-_{(2)}$	n.e.
T_5	3	0	2	0	0	$-_{(10)}$	$+_{(12)}$	$-_{(-1)}$	n.e.
T_6	0	3	1	0	0	$-_{(10)}$	$-_{(13)}$	$-_{(1)}$	n.e.
T_7	3	2	1	0	0	$-_{(10)}$	$+_{(12)}$	$-_{(-2)}$	n.e.
T_8	6	1	1	0	0	$+_{(9)}$	$+_{(11)}$	$+_{(-5)}$	$-1\frac{5}{8}$
T_9	9	0	1	0	0	$+_{(8)}$	$+_{(10)}$	$+_{(-8)}$	$-1,6$
T_{10}	0	5	0	0	0	$+_{(9)}$	$+_{(12)}$	$-_{(0)}$	n.e.
T_{11}	3	4	0	0	0	$+_{(9)}$	$+_{(12)}$	$+_{(-3)}$	$\mathbf{-1\frac{5}{7}}$
T_{12}	6	3	0	0	0	$+_{(9)}$	$+_{(11)}$	$+_{(-6)}$	$-1\frac{2}{3}$
T_{13}	9	2	0	0	0	$+_{(8)}$	$+_{(10)}$	$+_{(-9)}$	$-1\frac{7}{11}$
T_{14}	12	1	0	0	0	$+_{(7)}$	$+_{(9)}$	$+_{(-12)}$	$-1\frac{8}{13}$
T_{15}	15	0	0	0	0	$+_{(6)}$	$+_{(8)}$	$+_{(-15)}$	$-1,6$

4.6. Seven $\mathbb{R}$-lines

	t_2	t_3	t_4	t_5	t_6	t_7	$C_{1(\leq 10)}$	$C_{2(\leq 13)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_1	0	0	0	0	0	1	$+_t$	$+_t$	$+_t$	0
T_2	0	0	1	0	1	0	$+_t$	$+_t$	$-_{(4)}$	n.e.
T_3	0	2	0	0	1	0	$-_{(12)}$	$+_t$	$-_{(3)}$	n.e.
T_4	3	1	0	0	1	0	$-_{(11)}$	$+_{(13)}$	$-_{(0)}$	n.e.
T_5	6	0	0	0	1	0	$+_{(10)}$	$+_{(12)}$	$+_{(-3)}$	$-1\frac{4}{7}$
T_6	1	0	0	2	0	0	$-_{(11)}$	$+_t$	$-_{(3)}$	n.e.

	t_2	t_3	t_4	t_5	$C_{1(\leq 10)}$	$C_{2(\leq 13)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_7	2	1	1	1	$-_{(12)}$	$-_{(14)}$	$-_{(1)}$	n.e.
T_8	5	0	1	1	$-_{(11)}$	$+_{(13)}$	$-_{(-2)}$	n.e.
T_9	2	3	0	1	$-_{(11)}$	$-_{(14)}$	$-_{(0)}$	n.e.
T_{10}	5	2	0	1	$-_{(11)}$	$+_{(13)}$	$+_{(-3)}$	n.e.
T_{11}	8	1	0	1	$+_{(10)}$	$+_{(12)}$	$+_{(-6)}$	$-1,7$
T_{12}	11	0	0	1	$+_{(9)}$	$+_{(11)}$	$+_{(-9)}$	$-1\frac{2}{3}$
T_{13}	0	1	3	0	$-_{(12)}$	$-_{(15)}$	$-_{(3)}$	n.e.

	t_2	t_3	t_4	$C_{1(\leq 10)}$	$C_{2(\leq 13)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_{14}	3	0	3	$-_{(12)}$	$-_{(14)}$	$-_{(0)}$	n.e.
T_{15}	0	3	2	$-_{(11)}$	$-_{(14)}$	$-_{(2)}$	n.e.
T_{16}	3	2	2	$-_{(11)}$	$-_{(14)}$	$-_{(-1)}$	n.e.
T_{17}	6	1	2	$-_{(11)}$	$+_{(13)}$	$+_{(-4)}$	n.e.
T_{18}	9	0	2	$+_{(10)}$	$+_{(12)}$	$+_{(-7)}$	$-1\frac{8}{11}$
T_{19}	0	5	1	$+_{(10)}$	$+_{(13)}$	$-_{(1)}$	n.e.
T_{20}	3	4	1	$+_{(10)}$	$+_{(13)}$	$-_{(-2)}$	n.e.
T_{21}	6	3	1	$+_{(10)}$	$+_{(13)}$	$+_{(-5)}$	$-1,8$
T_{22}	9	2	1	$+_{(10)}$	$+_{(12)}$	$+_{(-8)}$	$-1\frac{3}{4}$
T_{23}	12	1	1	$+_{(9)}$	$+_{(11)}$	$+_{(-11)}$	$-1\frac{5}{7}$
T_{24}	15	0	1	$+_{(8)}$	$+_{(10)}$	$+_{(-14)}$	$-1\frac{11}{16}$
T_{25}	0	7	0	$+_{(9)}$	$+_{(12)}$	$-_{(0)}$	n.e.
T_{26}	3	6	0	$+_{(9)}$	$+_{(12)}$	$+_{(-3)}$	$\mathbf{-1\frac{8}{9}}$
T_{27}	6	5	0	$+_{(9)}$	$+_{(12)}$	$+_{(-6)}$	$-1\frac{9}{11}$
T_{28}	9	4	0	$+_{(9)}$	$+_{(12)}$	$+_{(-9)}$	$-1\frac{10}{13}$
T_{29}	12	3	0	$+_{(9)}$	$+_{(11)}$	$+_{(-12)}$	$-1\frac{11}{15}$
T_{30}	15	2	0	$+_{(8)}$	$+_{(10)}$	$+_{(-15)}$	$-1\frac{12}{17}$
T_{31}	18	1	0	$+_{(7)}$	$+_{(9)}$	$+_{(-18)}$	$-1\frac{13}{19}$
T_{32}	21	0	0	$+_{(6)}$	$+_{(8)}$	$+_{(-21)}$	$-1\frac{2}{3}$

4.7. Eight $\mathbb{R}$-lines

	t_2	t_3	t_4	t_5	t_6	t_7	t_8	$C_{1(\leq 11)}$	$C_{2(\leq 14)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_1	0	0	0	0	0	0	1	$+_t$	$+_t$	$+_t$	0

	t_2	t_3	t_4	t_5	t_6	t_7	t_8	$C_{1(\leq 11)}$	$C_{2(\leq 14)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_2	1	0	1	0	0	1	0	$-_{(13)}$	$+_t$	$-_{(4)}$	$n.e.$
T_3	1	2	0	0	0	1	0	$-_{(13)}$	$+_t$	$-_{(3)}$	$n.e.$
T_4	4	1	0	0	0	1	0	$-_{(13)}$	$-_{(15)}$	$-_{(0)}$	$n.e.$
T_5	7	0	0	0	0	1	0	$+_{(11)}$	$+_{(13)}$	$+_{(-3)}$	$-1\frac{5}{8}$
T_6	0	1	0	1	1	0	0	$-_{(14)}$	$+_t$	$-_{(5)}$	$n.e.$

	t_2	t_3	t_4	t_5	t_6	$C_{1(\leq 11)}$	$C_{2(\leq 14)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_7	3	0	0	1	1	$-_{(13)}$	$-_{(15)}$	$-_{(2)}$	$n.e.$
T_8	1	0	2	0	1	$-_{(14)}$	$-_{(16)}$	$-_{(4)}$	$n.e.$
T_9	1	2	1	0	1	$-_{(13)}$	$-_{(16)}$	$-_{(0)}$	$n.e.$
T_{10}	4	1	1	0	1	$-_{(13)}$	$-_{(15)}$	$-_{(0)}$	$n.e.$
T_{11}	7	0	1	0	1	$-_{(12)}$	$+_{(14)}$	$+_{(-3)}$	$n.e.$
T_{12}	1	4	0	0	1	$-_{(12)}$	$-_{(15)}$	$-_{(2)}$	$n.e.$
T_{13}	4	3	0	0	1	$-_{(12)}$	$-_{(15)}$	$-_{(-1)}$	$n.e.$
T_{14}	7	2	0	0	1	$-_{(12)}$	$+_{(14)}$	$+_{(-4)}$	$n.e.$
T_{15}	10	1	0	0	1	$+_{(11)}$	$+_{(13)}$	$+_{(-7)}$	$-1\frac{3}{4}$
T_{16}	13	0	0	0	1	$+_{(10)}$	$+_{(12)}$	$+_{(-10)}$	$-1\frac{5}{7}$
T_{17}	2	0	1	2	0	$-_{(14)}$	$-_{(16)}$	$-_{(3)}$	$n.e.$

	t_2	t_3	t_4	t_5	$C_{1(\leq 11)}$	$C_{2(\leq 14)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_{18}	2	2	0	2	$-_{(13)}$	$-_{(16)}$	$-_{(2)}$	$n.e.$
T_{19}	5	1	0	2	$-_{(13)}$	$-_{(15)}$	$-_{(-1)}$	$n.e.$
T_{20}	8	0	0	2	$-_{(12)}$	$+_{(14)}$	$+_{(-4)}$	$n.e.$
T_{21}	0	0	3	1	$-_{(13)}$	$-_{(17)}$	$-_{(5)}$	$n.e.$
T_{22}	0	2	2	1	$-_{(13)}$	$-_{(16)}$	$-_{(4)}$	$n.e.$
T_{23}	3	1	2	1	$-_{(13)}$	$-_{(16)}$	$-_{(1)}$	$n.e.$
T_{24}	6	0	2	1	$-_{(13)}$	$-_{(15)}$	$-_{(-2)}$	$n.e.$
T_{25}	0	4	1	1	$-_{(12)}$	$-_{(15)}$	$-_{(3)}$	$n.e.$
T_{26}	3	3	1	1	$-_{(12)}$	$-_{(15)}$	$-_{(0)}$	$n.e.$
T_{27}	6	2	1	1	$-_{(12)}$	$-_{(15)}$	$+_{(-3)}$	$n.e.$
T_{28}	9	1	1	1	$-_{(12)}$	$+_{(14)}$	$+_{(-6)}$	$n.e.$
T_{29}	12	0	1	1	$+_{(11)}$	$+_{(13)}$	$+_{(-9)}$	$-1\frac{11}{14}$
T_{30}	0	6	0	1	$+_{(11)}$	$+_{(14)}$	$-_{(2)}$	$n.e.$
T_{31}	3	5	0	1	$+_{(11)}$	$+_{(14)}$	$-_{(-1)}$	$n.e.$
T_{32}	6	4	0	1	$+_{(11)}$	$+_{(14)}$	$+_{(-4)}$	**n.e.***
T_{33}	9	3	0	1	$+_{(11)}$	$+_{(14)}$	$+_{(-7)}$	$-1\frac{11}{13}$
T_{34}	12	2	0	1	$+_{(11)}$	$+_{(13)}$	$+_{(-10)}$	$-1\frac{4}{5}$
T_{35}	15	1	0	1	$+_{(10)}$	$+_{(12)}$	$+_{(-13)}$	$-1\frac{13}{17}$
T_{36}	18	0	0	1	$+_{(9)}$	$+_{(11)}$	$+_{(-16)}$	$-1\frac{14}{19}$
T_{37}	1	1	4	0	$-_{(12)}$	$-_{(16)}$	$-_{(3)}$	$n.e.$

	t_2	t_3	t_4	$C_{1(\leq 11)}$	$C_{2(\leq 14)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_{38}	4	0	4	$-_{(12)}$	$-_{(16)}$	$-_{(0)}$	$n.e.$
T_{39}	1	3	3	$-_{(12)}$	$-_{(15)}$	$-_{(2)}$	$n.e.$
T_{40}	4	2	3	$-_{(12)}$	$-_{(15)}$	$-_{(-1)}$	$n.e.$

	t_2	t_3	t_4	$C_{1(\leq 11)}$	$C_{2(\leq 14)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_{41}	7	1	3	$-_{(12)}$	$-_{(15)}$	$+_{(-4)}$	$n.e.$
T_{42}	10	0	3	$-_{(12)}$	$+_{(14)}$	$+_{(-7)}$	$n.e.$
T_{43}	1	5	2	$+_{(11)}$	$+_{(14)}$	$-_{(1)}$	$n.e.$
T_{44}	4	4	2	$+_{(11)}$	$+_{(14)}$	$-_{(-2)}$	$n.e.$
T_{45}	7	3	2	$+_{(11)}$	$+_{(14)}$	$+_{(-5)}$	$-1\frac{11}{12}$
T_{46}	10	2	2	$+_{(11)}$	$+_{(14)}$	$+_{(-8)}$	$-1\frac{6}{7}$
T_{47}	13	1	2	$+_{(11)}$	$+_{(13)}$	$+_{(-11)}$	$-1\frac{13}{16}$
T_{48}	16	0	2	$+_{(10)}$	$+_{(12)}$	$+_{(-14)}$	$-1\frac{7}{9}$
T_{49}	1	7	1	$+_{(10)}$	$+_{(13)}$	$-_{(0)}$	$n.e.$
T_{50}	4	6	1	$+_{(10)}$	$+_{(13)}$	$+_{(-3)}$	$\mathbf{-2}$
T_{51}	7	5	1	$+_{(10)}$	$+_{(13)}$	$+_{(-6)}$	$-1\frac{12}{13}$
T_{52}	10	4	1	$+_{(10)}$	$+_{(13)}$	$+_{(-9)}$	$-1\frac{13}{15}$
T_{53}	13	3	1	$+_{(10)}$	$+_{(13)}$	$+_{(-12)}$	$-1\frac{14}{17}$
T_{54}	16	2	1	$+_{(10)}$	$+_{(12)}$	$+_{(-15)}$	$-1\frac{15}{19}$
T_{55}	19	1	1	$+_{(9)}$	$+_{(11)}$	$+_{(-18)}$	$-1\frac{16}{21}$
T_{56}	22	0	1	$+_{(8)}$	$+_{(10)}$	$+_{(-21)}$	$-1\frac{17}{23}$
T_{57}	1	9	0	$+_{(9)}$	$+_{(12)}$	$-_{(-1)}$	$n.e.$
T_{58}	4	8	0	$+_{(9)}$	$+_{(12)}$	$+_{(-4)}$	**n.e.** **
T_{59}	7	7	0	$+_{(9)}$	$+_{(12)}$	$+_{(-7)}$	$-1\frac{13}{14}$
T_{60}	10	6	0	$+_{(9)}$	$+_{(12)}$	$+_{(-10)}$	$-1\frac{7}{8}$
T_{61}	13	5	0	$+_{(9)}$	$+_{(12)}$	$+_{(-13)}$	$-1\frac{5}{6}$
T_{62}	16	4	0	$+_{(9)}$	$+_{(12)}$	$+_{(-16)}$	$-1\frac{4}{5}$
T_{63}	19	3	0	$+_{(9)}$	$+_{(11)}$	$+_{(-19)}$	$-1\frac{17}{22}$
T_{64}	22	2	0	$+_{(8)}$	$+_{(10)}$	$+_{(-22)}$	$-1\frac{3}{4}$
T_{65}	25	1	0	$+_{(7)}$	$+_{(9)}$	$+_{(-25)}$	$-1\frac{19}{26}$
T_{66}	28	0	0	$+_{(6)}$	$+_{(8)}$	$+_{(-28)}$	$-1\frac{5}{7}$

* Solution T_{32} is excluded by Criterion C4.

** Solution T_{58} is not excluded by any criterion C1-C4, but it gives the so-called Möbius-Kantor configuration which cannot be drawn in the real projective plane (see [7], Figure 9).

Therefore, for $d \geqslant 8$ the set of criteria {C1,C2,C3,C4} is not enough.

4.8. Nine $\mathbb{R}$-lines

	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	$C_{1(\leq 12)}$	$C_{2(\leq 15)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_1	0	0	0	0	0	0	0	1	$+_t$	$+_t$	$+_t$	0
T_2	2	0	1	0	0	0	1	0	$-_{(14)}$	$-_{(16)}$	$-_{(4)}$	$n.e.$
T_3	2	2	0	0	0	0	1	0	$-_{(14)}$	$-_{(16)}$	$-_{(3)}$	$n.e.$
T_4	5	1	0	0	0	0	1	0	$-_{(13)}$	$+_{(15)}$	$-_{(0)}$	$n.e.$
T_5	8	0	0	0	0	0	1	0	$+_{(12)}$	$+_{(14)}$	$+_{(-3)}$	$-1\frac{2}{3}$
T_6	0	0	0	0	1	1	0	0	$+_t$	$+_t$	$-_{(7)}$	$n.e.$

	t_2	t_3	t_4	t_5	t_6	t_7	$C_{1(\leq 12)}$	$C_{2(\leq 15)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_7	2	1	0	1	0	1	$-_{(15)}$	$-_{(17)}$	$-_{(4)}$	$n.e.$
T_8	5	0	0	1	0	1	$-_{(14)}$	$-_{(16)}$	$-_{(1)}$	$n.e.$
T_9	0	1	2	0	0	1	$-_{(15)}$	$-_{(18)}$	$-_{(6)}$	$n.e.$
T_{10}	3	0	2	0	0	1	$-_{(15)}$	$-_{(17)}$	$-_{(3)}$	$n.e.$

	t_2	t_3	t_4	t_5	t_6	t_7	$C_{1(\leq 12)}$	$C_{2(\leq 15)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_{11}	0	3	1	0	0	1	$-_{(14)}$	$-_{(17)}$	$-_{(5)}$	$n.e.$
T_{12}	3	2	1	0	0	1	$-_{(14)}$	$-_{(17)}$	$-_{(2)}$	$n.e.$
T_{13}	6	1	1	0	0	1	$-_{(14)}$	$-_{(16)}$	$-_{(-1)}$	$n.e.$
T_{14}	9	0	1	0	0	1	$-_{(13)}$	$+_{(15)}$	$+_{(-4)}$	$n.e.$
T_{15}	0	5	0	0	0	1	$-_{(13)}$	$-_{(16)}$	$-_{(4)}$	$n.e.$
T_{16}	3	4	0	0	0	1	$-_{(13)}$	$-_{(16)}$	$-_{(1)}$	$n.e.$
T_{17}	6	3	0	0	0	1	$-_{(13)}$	$-_{(16)}$	$-_{(-2)}$	$n.e.$
T_{18}	9	2	0	0	0	1	$-_{(13)}$	$+_{(15)}$	$+_{(-5)}$	$n.e.$
T_{19}	12	1	0	0	0	1	$+_{(12)}$	$+_{(14)}$	$+_{(-8)}$	$-1\frac{11}{14}$
T_{20}	15	0	0	0	0	1	$+_{(11)}$	$+_{(13)}$	$+_{(-11)}$	$-1\frac{3}{4}$
T_{21}	0	0	1	0	2	0	$-_{(16)}$	$+_{t}$	$-_{(7)}$	$n.e.$

	t_2	t_3	t_4	t_5	t_6	$C_{1(\leq 12)}$	$C_{2(\leq 15)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_{22}	0	2	0	0	2	$-_{(15)}$	$-_{(18)}$	$-_{(6)}$	$n.e.$
T_{23}	3	1	0	0	2	$-_{(15)}$	$-_{(17)}$	$-_{(3)}$	$n.e.$
T_{24}	6	0	0	0	2	$-_{(14)}$	$-_{(16)}$	$-_{(0)}$	$n.e.$
T_{25}	1	0	0	2	1	$-_{(16)}$	$-_{(18)}$	$-_{(6)}$	$n.e.$
T_{26}	2	1	1	1	1	$-_{(15)}$	$-_{(18)}$	$-_{(4)}$	$n.e.$
T_{27}	5	0	1	1	1	$-_{(15)}$	$-_{(17)}$	$-_{(1)}$	$n.e.$
T_{28}	2	3	0	1	1	$-_{(14)}$	$-_{(17)}$	$-_{(3)}$	$n.e.$
T_{29}	5	2	0	1	1	$-_{(14)}$	$-_{(17)}$	$-_{(0)}$	$n.e.$
T_{30}	8	1	0	1	1	$-_{(14)}$	$-_{(16)}$	$+_{(-3)}$	$n.e.$
T_{31}	11	0	0	1	1	$-_{(13)}$	$+_{(15)}$	$+_{(-6)}$	$n.e.$
T_{32}	0	1	3	0	1	$-_{(14)}$	$-_{(18)}$	$-_{(6)}$	$n.e.$
T_{33}	3	0	3	0	1	$-_{(14)}$	$-_{(18)}$	$-_{(3)}$	$n.e.$
T_{34}	0	3	2	0	1	$-_{(14)}$	$-_{(17)}$	$-_{(5)}$	$n.e.$
T_{35}	3	2	2	0	1	$-_{(14)}$	$-_{(17)}$	$-_{(2)}$	$n.e.$
T_{36}	6	1	2	0	1	$-_{(14)}$	$-_{(17)}$	$-_{(-1)}$	$n.e.$
T_{37}	9	0	2	0	1	$-_{(14)}$	$-_{(16)}$	$+_{(-4)}$	$n.e.$
T_{38}	0	5	1	0	1	$-_{(13)}$	$-_{(16)}$	$-_{(4)}$	$n.e.$
T_{39}	3	4	1	0	1	$-_{(13)}$	$-_{(16)}$	$-_{(1)}$	$n.e.$
T_{40}	6	3	1	0	1	$-_{(13)}$	$-_{(16)}$	$-_{(-2)}$	$n.e.$
T_{41}	9	2	1	0	1	$-_{(13)}$	$-_{(16)}$	$+_{(-5)}$	$n.e.$
T_{42}	12	1	1	0	1	$-_{(13)}$	$+_{(15)}$	$+_{(-8)}$	$n.e.$
T_{43}	15	0	1	0	1	$+_{(12)}$	$+_{(14)}$	$+_{(-11)}$	$-1\frac{14}{17}$
T_{44}	0	7	0	0	1	$+_{(12)}$	$+_{(15)}$	$-_{(3)}$	$n.e.$
T_{45}	3	6	0	0	1	$+_{(12)}$	$+_{(15)}$	$-_{(0)}$	$n.e.$
T_{46}	6	5	0	0	1	$+_{(12)}$	$+_{(15)}$	$+_{(-3)}$	**n.e.** ***
T_{47}	9	4	0	0	1	$+_{(12)}$	$+_{(15)}$	$+_{(-6)}$	**n.e.** ***

	t_2	t_3	t_4	t_5	t_6	$C_{1(\leq 12)}$	$C_{2(\leq 15)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_{48}	12	3	0	0	1	$+_{(12)}$	$+_{(15)}$	$+_{(-9)}$	$-1\frac{7}{8}$
T_{49}	15	2	0	0	1	$+_{(12)}$	$+_{(14)}$	$+_{(-12)}$	$-1\frac{5}{6}$
T_{50}	18	1	0	0	1	$+_{(11)}$	$+_{(13)}$	$+_{(-15)}$	$-1\frac{4}{5}$
T_{51}	21	0	0	0	1	$+_{(10)}$	$+_{(12)}$	$+_{(-18)}$	$-1\frac{17}{22}$
T_{52}	0	0	1	3	0	$-_{(15)}$	$-_{(19)}$	$-_{(7)}$	*n.e.*

	t_2	t_3	t_4	t_5	$C_{1(\leq 12)}$	$C_{2(\leq 15)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_{53}	0	2	0	3	$-_{(15)}$	$-_{(18)}$	$-_{(6)}$	*n.e.*
T_{54}	3	1	0	3	$-_{(15)}$	$-_{(18)}$	$-_{(3)}$	*n.e.*
T_{55}	6	0	0	3	$-_{(15)}$	$-_{(17)}$	$-_{(0)}$	*n.e.*
T_{56}	1	1	2	2	$-_{(14)}$	$-_{(18)}$	$-_{(5)}$	*n.e.*
T_{57}	4	0	2	2	$-_{(14)}$	$-_{(18)}$	$-_{(2)}$	*n.e.*
T_{58}	1	3	1	2	$-_{(14)}$	$-_{(17)}$	$-_{(4)}$	*n.e.*
T_{59}	4	2	1	2	$-_{(14)}$	$-_{(17)}$	$-_{(1)}$	*n.e.*
T_{60}	7	1	1	2	$-_{(14)}$	$-_{(17)}$	$-_{(-2)}$	*n.e.*
T_{61}	10	0	1	2	$-_{(14)}$	$-_{(16)}$	$+_{(-5)}$	*n.e.*
T_{62}	1	5	0	2	$-_{(13)}$	$-_{(16)}$	$-_{(3)}$	*n.e.*
T_{63}	4	4	0	2	$-_{(13)}$	$-_{(16)}$	$-_{(0)}$	*n.e.*
T_{64}	7	3	0	2	$-_{(13)}$	$-_{(16)}$	$+_{(-3)}$	*n.e.*
T_{65}	10	2	0	2	$-_{(13)}$	$-_{(16)}$	$+_{(-6)}$	*n.e.*
T_{66}	13	1	0	2	$-_{(13)}$	$+_{(15)}$	$+_{(-9)}$	*n.e.*
T_{67}	16	0	0	2	$+_{(12)}$	$+_{(14)}$	$+_{(-12)}$	$-1\frac{5}{6}$
T_{68}	2	0	4	1	$-_{(13)}$	$-_{(17)}$	$-_{(4)}$	*n.e.*
T_{69}	2	2	3	1	$-_{(13)}$	$-_{(17)}$	$-_{(3)}$	*n.e.*
T_{70}	5	1	3	1	$-_{(13)}$	$-_{(17)}$	$-_{(0)}$	*n.e.*
T_{71}	8	0	3	1	$-_{(13)}$	$-_{(17)}$	$+_{(-3)}$	*n.e.*
T_{72}	2	4	2	1	$-_{(13)}$	$-_{(16)}$	$-_{(2)}$	*n.e.*
T_{73}	5	3	2	1	$-_{(13)}$	$-_{(16)}$	$-_{(-1)}$	*n.e.*
T_{74}	8	2	2	1	$-_{(13)}$	$-_{(16)}$	$+_{(-4)}$	*n.e.*
T_{75}	11	1	2	1	$-_{(13)}$	$-_{(16)}$	$+_{(-7)}$	*n.e.*
T_{76}	14	0	2	1	$-_{(13)}$	$+_{(15)}$	$+_{(-10)}$	*n.e.*
T_{77}	2	6	1	1	$+_{(12)}$	$+_{(15)}$	$-_{(1)}$	*n.e.*
T_{78}	5	5	1	1	$+_{(12)}$	$+_{(15)}$	$-_{(-2)}$	*n.e.*
T_{79}	8	4	1	1	$+_{(12)}$	$+_{(15)}$	$+_{(-5)}$	**n.e.**$^{***}_{*}$
T_{80}	11	3	1	1	$+_{(12)}$	$+_{(15)}$	$+_{(-8)}$	$-1\frac{15}{16}$
T_{81}	14	2	1	1	$+_{(12)}$	$+_{(15)}$	$+_{(-11)}$	$-1\frac{8}{9}$
T_{82}	17	1	1	1	$+_{(12)}$	$+_{(14)}$	$+_{(-14)}$	$-1\frac{17}{20}$
T_{83}	20	0	1	1	$+_{(11)}$	$+_{(13)}$	$+_{(-17)}$	$-1\frac{9}{11}$
T_{84}	2	8	0	1	$+_{(11)}$	$+_{(14)}$	$-_{(0)}$	*n.e.*
T_{85}	5	7	0	1	$+_{(11)}$	$+_{(14)}$	$+_{(-3)}$	**n.e.**$^{***}_{**}$
T_{86}	8	6	0	1	$+_{(11)}$	$+_{(14)}$	$+_{(-6)}$	-2
T_{87}	11	5	0	1	$+_{(11)}$	$+_{(14)}$	$+_{(-9)}$	$-1\frac{16}{17}$
T_{88}	14	4	0	1	$+_{(11)}$	$+_{(14)}$	$+_{(-12)}$	$-1\frac{17}{19}$
T_{89}	17	3	0	1	$+_{(11)}$	$+_{(14)}$	$+_{(-15)}$	$-1\frac{6}{7}$
T_{90}	20	2	0	1	$+_{(11)}$	$+_{(13)}$	$+_{(-18)}$	$-1\frac{19}{23}$

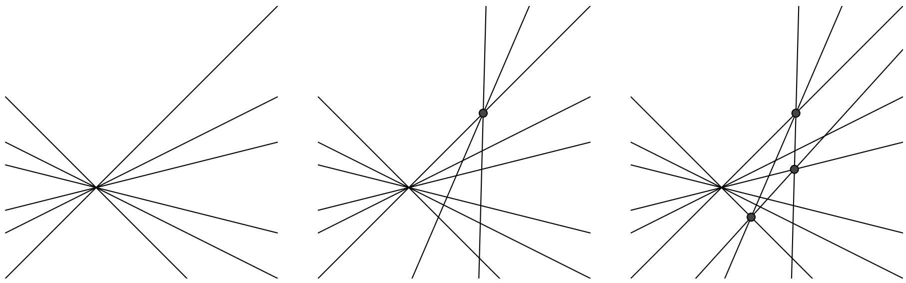
	t_2	t_3	t_4	t_5	$C_{1(\leq 12)}$	$C_{2(\leq 15)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_{91}	23	1	0	1	$+_{(10)}$	$+_{(12)}$	$+_{(-21)}$	$-1\frac{4}{5}$
T_{92}	26	0	0	1	$+_{(9)}$	$+_{(11)}$	$+_{(-24)}$	$-1\frac{7}{9}$
T_{93}	0	0	6	0	$+_{(12)}$	$-_{(16)}$	$-_{(6)}$	$n.e.$

	t_2	t_3	t_4	$C_{1(\leq 12)}$	$C_{2(\leq 15)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_{94}	0	2	5	$+_{(12)}$	$-_{(16)}$	$-_{(5)}$	$n.e.$
T_{95}	3	1	5	$+_{(12)}$	$-_{(16)}$	$-_{(2)}$	$n.e.$
T_{96}	6	0	5	$+_{(12)}$	$-_{(16)}$	$-_{(-1)}$	$n.e.$
T_{97}	0	4	4	$+_{(12)}$	$-_{(16)}$	$-_{(4)}$	$n.e.$
T_{98}	3	3	4	$+_{(12)}$	$-_{(16)}$	$-_{(1)}$	$n.e.$
T_{99}	6	2	4	$+_{(12)}$	$-_{(16)}$	$-_{(-2)}$	$n.e.$
T_{100}	9	1	4	$+_{(12)}$	$-_{(16)}$	$+_{(-5)}$	$n.e.$
T_{101}	12	0	4	$+_{(12)}$	$-_{(16)}$	$+_{(-8)}$	$n.e.$
T_{102}	0	6	3	$+_{(12)}$	$+_{(15)}$	$-_{(3)}$	$n.e.$
T_{103}	3	5	3	$+_{(12)}$	$+_{(15)}$	$-_{(0)}$	$n.e.$
T_{104}	6	4	3	$+_{(12)}$	$+_{(15)}$	$+_{(-3)}$	$\mathbf{-2\frac{1}{13}}$
T_{105}	9	3	3	$+_{(12)}$	$+_{(15)}$	$+_{(-6)}$	-2
T_{106}	12	2	3	$+_{(12)}$	$+_{(15)}$	$+_{(-9)}$	$-1\frac{16}{17}$
T_{107}	15	1	3	$+_{(12)}$	$+_{(15)}$	$+_{(-12)}$	$-1\frac{17}{19}$
T_{108}	18	0	3	$+_{(12)}$	$+_{(14)}$	$+_{(-15)}$	$-1\frac{6}{7}$
T_{109}	0	8	2	$+_{(11)}$	$+_{(14)}$	$-_{(2)}$	$n.e.$
T_{110}	3	7	2	$+_{(11)}$	$+_{(14)}$	$-_{(-1)}$	$n.e.$
T_{111}	6	6	2	$+_{(11)}$	$+_{(14)}$	$+_{(-4)}$	$-2\frac{1}{14}$
T_{112}	9	5	2	$+_{(11)}$	$+_{(14)}$	$+_{(-7)}$	-2
T_{113}	12	4	2	$+_{(11)}$	$+_{(14)}$	$+_{(-10)}$	$-1\frac{17}{18}$
T_{114}	15	3	2	$+_{(11)}$	$+_{(14)}$	$+_{(-13)}$	$-1\frac{9}{10}$
T_{115}	18	2	2	$+_{(11)}$	$+_{(14)}$	$+_{(-16)}$	$-1\frac{19}{22}$
T_{116}	21	1	2	$+_{(11)}$	$+_{(13)}$	$+_{(-19)}$	$-1\frac{5}{6}$
T_{117}	24	0	2	$+_{(10)}$	$+_{(13)}$	$+_{(-22)}$	$-1\frac{21}{26}$
T_{118}	0	10	1	$+_{(10)}$	$+_{(13)}$	$-_{(1)}$	$n.e.$
T_{119}	3	9	1	$+_{(10)}$	$+_{(13)}$	$-_{(-2)}$	$n.e.$
T_{120}	6	8	1	$+_{(10)}$	$+_{(13)}$	$+_{(-5)}$	$-2\frac{1}{15}$
T_{121}	9	7	1	$+_{(10)}$	$+_{(13)}$	$+_{(-8)}$	-2
T_{122}	12	6	1	$+_{(10)}$	$+_{(13)}$	$+_{(-11)}$	$-1\frac{18}{19}$
T_{123}	15	5	1	$+_{(10)}$	$+_{(13)}$	$+_{(-14)}$	$-1\frac{19}{21}$
T_{124}	18	4	1	$+_{(10)}$	$+_{(13)}$	$+_{(-17)}$	$-1\frac{20}{23}$
T_{125}	21	3	1	$+_{(10)}$	$+_{(13)}$	$+_{(-20)}$	$-1\frac{21}{25}$
T_{126}	24	2	1	$+_{(10)}$	$+_{(12)}$	$+_{(-23)}$	$-1\frac{22}{27}$
T_{127}	27	1	1	$+_{(9)}$	$+_{(11)}$	$+_{(-26)}$	$-1\frac{23}{29}$
T_{128}	30	0	1	$+_{(8)}$	$+_{(10)}$	$+_{(-29)}$	$-1\frac{24}{31}$
T_{129}	0	12	0	$+_{(9)}$	$+_{(12)}$	$-_{(0)}$	$n.e.$
T_{130}	3	11	0	$+_{(9)}$	$+_{(12)}$	$+_{(-3)}$	**n.e.**$^{***}_{***}$
T_{131}	6	10	0	$+_{(9)}$	$+_{(12)}$	$+_{(-6)}$	$-2\frac{1}{16}$
T_{132}	9	9	0	$+_{(9)}$	$+_{(12)}$	$+_{(-9)}$	-2
T_{133}	12	8	0	$+_{(9)}$	$+_{(12)}$	$+_{(-12)}$	$-1\frac{19}{20}$

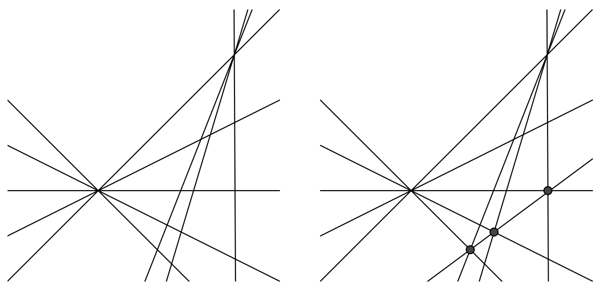
	t_2	t_3	t_4	$C_{1(\leq 12)}$	$C_{2(\leq 15)}$	$C_{3(\leq -3)}$	$q(T_i)$
T_{134}	15	7	0	$+_{(9)}$	$+_{(12)}$	$+_{(-15)}$	$-1\frac{10}{11}$
T_{135}	18	6	0	$+_{(9)}$	$+_{(12)}$	$+_{(-18)}$	$-1\frac{7}{8}$
T_{136}	21	5	0	$+_{(9)}$	$+_{(12)}$	$+_{(-21)}$	$-1\frac{11}{13}$
T_{137}	24	4	0	$+_{(9)}$	$+_{(12)}$	$+_{(-24)}$	$-1\frac{23}{28}$
T_{138}	27	3	0	$+_{(9)}$	$+_{(11)}$	$+_{(-27)}$	$-1\frac{4}{5}$
T_{139}	30	2	0	$+_{(8)}$	$+_{(10)}$	$+_{(-30)}$	$-1\frac{25}{32}$
T_{140}	33	1	0	$+_{(7)}$	$+_{(9)}$	$+_{(-33)}$	$-1\frac{13}{17}$
T_{141}	36	0	0	$+_{(6)}$	$+_{(8)}$	$+_{(-36)}$	$-1\frac{3}{4}$

*** Solution T_{46} is excluted by Criterion C4; solution T_{47} is not.

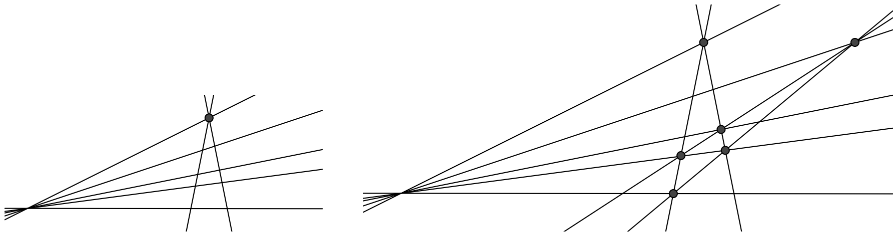
At the beginning, we construct a pencil of 6 $\mathbb{R}$-lines because $t_6 = 1$. Then we construct one point with multiplicity 3, using two other $\mathbb{R}$-lines. We can use only one more $\mathbb{R}$-line. The maximum number of triple points we can achieve is 3, as in the figure below.



**** At the beginning, we construct a pencil of 5 $\mathbb{R}$-lines, because $t_5 = 1$. Then we construct one point with multiplicity 4 (because $t_4 = 1$), using three other $\mathbb{R}$-lines. We can use only one more $\mathbb{R}$-line. The maximum number of triple points we can achieve is 3, as in the figure below.



***** At the beginning, we construct a pencil of 5 $\mathbb{R}$-lines because $t_5 = 1$. We want to maximize the number of triple points. We construct one point with multiplicity 3, using two other $\mathbb{R}$-lines. Two $\mathbb{R}$-lines are left. Using one of them, we can construct maximum two more triple points, using the second of them $-$ 3. Therefore, the maximum number of triple points we can achieve is 6, as in the figure below.



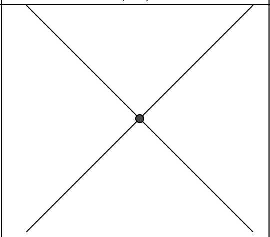
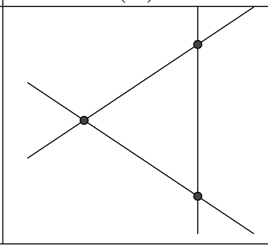
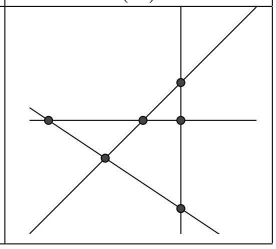
*** *** Extracting one of the lines from this configuration, it gives us Möbius-Kantor configuration, which cannot be realized over $\mathbb{R}$.

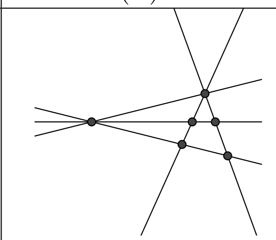
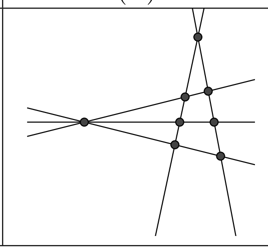
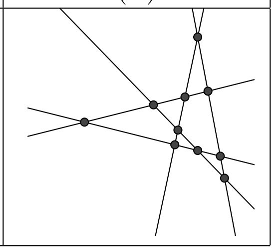
4.9. Main result

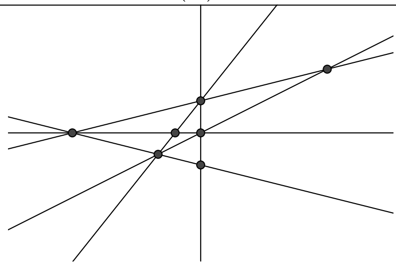
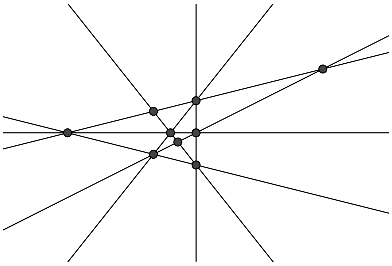
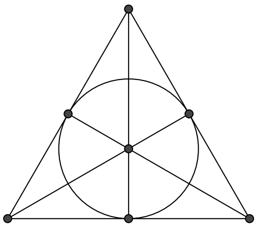
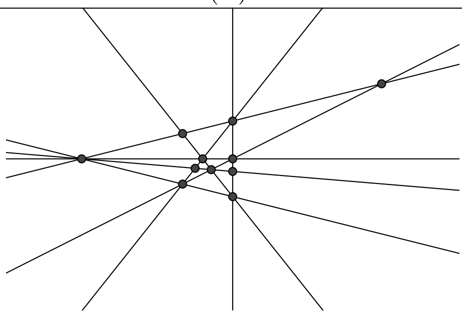
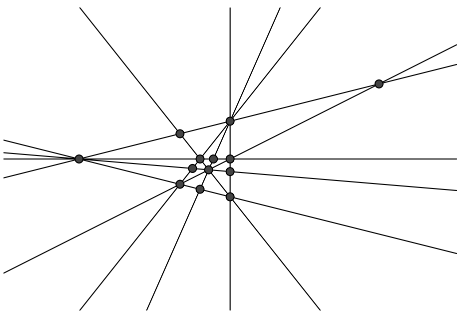
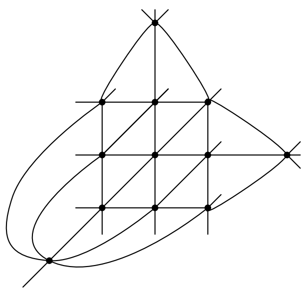
d	2	3	4	5	6	7	8	9	source
$\mathcal{H}_L(d)$	0	-1	$-1\frac{1}{3}$	$-1,5$	$-1\frac{5}{7}$	-2	-2	$\mathbf{-2\frac{1}{4} = -2,25}$	[7] [2]
$\mathcal{H}_L(\mathbb{C}, d)$	0	-1	$-1\frac{1}{3}$	$-1,5$	$-1\frac{5}{7}$	$-1\frac{8}{9}$	-2	$\mathbf{-2\frac{1}{4} = -2,25}$	[7]
$\mathcal{H}_L(\mathbb{R}, d)$	0	-1	$-1\frac{1}{3}$	$-1,5$	$-1\frac{5}{7}$	$-1\frac{8}{9}$	-2	$\mathbf{-2\frac{1}{13} \approx -2,077}$	own work

As we can see, we have $\mathcal{H}_L(\mathbb{C}, 7) = \mathcal{H}_L(\mathbb{R}, 7) \neq \mathcal{H}_L(7)$ (that means over other fields than $\mathbb{R}$ there exist arrangements computing the absolute Harbourne constant) and $\mathcal{H}_L(\mathbb{R}, 9) \neq \mathcal{H}_L(\mathbb{C}, 9)$ (that means over $\mathbb{C}$ there exists arrangement which cannot be realized over reals – e.g. dual Hesse configuration).

The lowest values of Harbourne constants are given by arrangements:

d	$(\mathbb{R})$	d	$(\mathbb{R})$	d	$(\mathbb{R})$
2		3		4	

d	$(\mathbb{R})$	$(\mathbb{R})$	$(\mathbb{R})$
5			

d	arrangements	
6	($\mathbb{R}$)	
		
7	($\mathbb{R}$)	($\mathbb{F}_2$) (Fano configuration)
		
8	($\mathbb{R}$)	
		
9	($\mathbb{R}$)	($\mathbb{C}$) (dual Hesse configuration)
		

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