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Magdalena Lampa-Baczyńska[1]

The effect of points fattening on the blow up of the projective plane at one point

Abstract. In this paper we study the points fattening effect over the complex numbers for the surface arising by blowing-up of $\mathbb{P}^2$ at one point. We denote this space by $\mathbb{S}_1$. This surface has been recently considered with respect to the points fattening, but as a Hirzebruch surface. We study this issue for $\mathbb{S}_1$ taken as del Pezzo surface. The choice of point of view for this space implies the choice of reference line bundle. We will show, among others, that the choice of the polarization is a fundamental factor affecting the shape of the initial sequence.

1. Introduction

The approach to fat point schemes has been initiated by Bocci and Chiantini in [3]. They, as the first, defined the initial degree $\alpha(I)$ of a homogeneous ideal $I \subset \mathbb{C}[\mathbb{P}^n]$ as the least degree t such that the homogeneous component I_t in degree t is non zero. The definition of the initial degree can be extended for any integer m, where vanishing along mZ means passing through points of Z with multiplicity m. This notion was first introduced by Chudnovsky in [5] but in another set-up and it was not given a name. Bocci and Chiantini for the first time used this invariant to study fat points subschemes in the projective plane. They proved, among other things, that zero dimensional subschemes Z of $\mathbb{P}^2$ such that

$$\alpha(2Z) - \alpha(Z) = 1,$$

i.e. such that the difference of the first two elements of the initial sequence is the minimal one, namely 1 in this case, are exactly the subschemes either contained in a single line or forming a so called star-configuration.

These considerations were extended for another types of spaces. Dumnicki, Szemberg and Tutaj-Gasińska in [7] were studying configurations of points in $\mathbb{P}^2$ with

$$\alpha((m+1)Z) - \alpha(mZ) = 1$$

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Słowa kluczowe: initial degree, initial sequence, Waldschmidt constant, blow-up, alpha problem.

for some $m \geq 2$ and obtained their full characterization.

From then on the alpha problem, called so because of the notation of the initial degree, was studied for some other spaces. For example results of Bocci and Chiantini were extended to the space $\mathbb{P}^3$ and in this context there was even formulated a conjecture for projective spaces of arbitrary dimension (see [2]). Except for spaces $\mathbb{P}^n$ recently the problem of points fattening was considered in [1] for the space $\mathbb{P}^1 \times \mathbb{P}^1$ and by Di Rocco, Lundman and Szemberg in [6] for Hirzebruch surfaces. In general we define the initial degree as follows (see [6], Definition 1.1).

Definition 1.1 (Initial degree)
Let X be a smooth projective variety with an ample line bundle L on X and let Z be a reduced subscheme of X defined by the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_Z$. For a positive integer m the initial degree (with respect to L) of the subscheme mZ is the integer

$$\alpha(mZ) := \min\left\{ d : \ H^0(X, dL \otimes \mathcal{I}_Z^{(m)}) \neq 0 \ \right\}.$$

The initial sequence (with respect to L) of a subscheme Z is the sequence

$$\alpha(Z), \alpha(2Z), \alpha(3Z), \ldots$$

The initial sequence is a subadditive and weakly growing sequence of positive integers, so in particular we may consider its asymptotic invariant, namely Waldschmidt constant (see [9] and [4], Lemma 2.3.1).

Definition 1.2 (Waldschmidt constant of a subscheme)
Keeping the notation from Definition 1.1 we define the Waldschmidt constant of Z (with respect to L) as the limit

$$\widehat{\alpha}(Z) := \lim_{m\to\infty} \frac{\alpha(mZ)}{m}.$$

The choice of line bundle L strictly depends on the considered variety X. The interesting phenomenon is fact, that the surface arising by blowing-up of $\mathbb{P}^2$ at one point can be considered from two distinct points of view, as a Hirzebruch surface and as a del Pezzo surface. The most natural choices of the reference line bundle in these two cases are different. Di Rocco, Lundman, and Szemberg proved in [6], that on the Hirzebruch surface $\mathbb{S}_1$ (denoted there by $\mathbb{F}_1$) with the line bundle $2H - E_1$ (the optimal bundle for this Hirzebruch surface) there does not exist any finite set Z, such that

$$\alpha(Z) = \alpha(2Z) = \alpha(3Z) = \alpha(4Z)$$

(see [6], Proposition 4.1). In this paper we consider the surface $\mathbb{S}_1$ from point of view of the del Pezzo surfaces and we prove, that the choice of the polarization is a fundamental factor affecting the shape of the initial sequence.

The main results of this paper are full characterizations of sets $Z \subset \mathbb{S}_1$ satisfying the condition

$$\alpha(mZ) = \ldots = \alpha((m+t)Z)$$

with $t = 4$ and $t = 3$. These characterizations are given in Theorems 3.1 and 3.5. In Theorem 3.9 we also made description of this type for sets satisfying the weaker condition, namely $\alpha(Z) = \alpha(2Z) = \alpha(3Z)$. Let then pass to the details.

2. Blow-up of $\mathbb{P}^2$ at one point as a del Pezzo surface

In fact over the complex numbers, there are exactly 10 families of del Pezzo surfaces, including 8 arising by blowing up $\mathbb{P}^2$. They are blow ups in $1 \leq r \leq 8$ generic points, denoted by us as $\mathbb{S}_r$. Let us denote by $f_r : \mathbb{S}_r \to \mathbb{P}^2$ these blow ups and by $P_1, \ldots, P_r$ the points blown up. The $E_1, \ldots, E_r$ are the exceptional divisors over these points. In our case r is fixed, i.e. $r = 1$ then we write simply f instead of f_1. As the reference ample line bundle on del Pezzo surfaces $\mathbb{S}_r$ we take the anticanonical bundle

$$\mathbb{L}_r = -K_{\mathbb{S}_r} = 3H - E_1 - \ldots - E_r,$$

which is not divisible in the Picard group $Pic(\mathbb{S}_r)$. This seems to be the most natural choice in this case. Thus for surface $\mathbb{S}_1$ we work with the bundle $3H - E_1$.

To understand some of our considerations better, we present a few schematic pictures illustrating the behaviour of some plane curves after blowing up the plane in a fixed point. It is a little bit complicated to make exact graphic presentation of a total transform of any curve (especially that we work over $\mathbb{C}$). It is so even in the case of the blowing up of a single point. For a greater number of points such an exact and detailed picture may not be possible to make or it would be confusing and not transparent. For that reason our pictures are simplified. In Figures 1 and 2 we present an example of such a simplified schematic picture compared to a detailed graphic presentation of the total transform of a line in the blow up at one point.

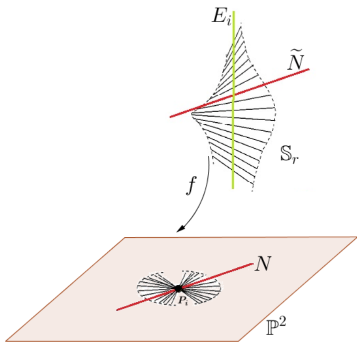

Figure 1: Detailed picture

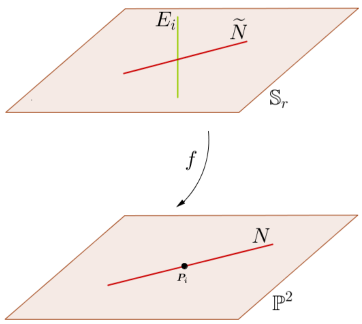

Figure 2: Schematic picture

In further considerations we will use the following observation about blow ups.

Remark 2.1
If F is a plane curve of degree $3k$ in $\mathbb{P}^2$ passing through the points $P_1, \ldots, P_r$, so that $\text{mult}_{P_i}(F) = m_i \geq k$ for $i \in \{1, \ldots, r\}$, then E_i is a $(m_i - k)$−tuple component of the divisor $f^*(F) - kE_1 - \ldots - kE_r$ in the system $|3kH - kE_1 - \ldots - kE_r|$ on $\mathbb{S}_r$.

Definition 2.2 (Adapted transform)
We keep the notation from Remark 2.1. The divisor

$$f_a^* F := f_r^*(F) - kE_1 - \ldots - kE_r = \widetilde{F} + \sum\nolimits_{i=1}^{r}(m_i - k)E_i$$

is called the adapted transform of F.

Lemma 2.3
Let $D \in |k(3H - E_1 - \cdots - E_r)|$ for fixed $1 \leq r \leq 8$ and $Q \in \mathbb{S}_r$. Then

$$\mathrm{mult}_Q(D) \leq 2 \cdot \mathrm{mult}_{f_r(Q)}(f_r(D)) - k, \tag{2.1}$$

if $Q \in E_1 \cup \ldots \cup E_r$ and

$$\mathrm{mult}_Q(D) = \mathrm{mult}_{f_r(Q)}(f_r(D)) \leq 3k, \tag{2.2}$$

if $Q \notin E_1 \cup \ldots \cup E_r$. Furthermore, if equality holds in (2.2), then $f_r(D)$ is a union of lines through $f_r(Q)$.

Proof. Let $D \in |k(3H - E_1 - \cdots - E_r)|$ and $Q \in \mathbb{S}_r$. Then $\deg(f_r(D)) = 3k$. Let us denote by $m = \mathrm{mult}_Q(D)$.

First we consider the situation, when $Q \notin E_1 \cup \ldots \cup E_r$. Since f_r is an isomorphism away of points $\{P_1, \ldots, P_r\}$, then $\mathrm{mult}_{f_r(Q)}(f_r(D)) = m$. The multiplicity of the singular point of the plane curve can be at most the degree of this curve, thus $f_r(D)$ may have at most $3k-$tuple points, what finishes the proof of statement (2.2).

We assume now, that $Q \in E_i$ for some $i \in \{1, \ldots, r\}$. Let us denote by $F = f_r(D)$. Then

$$\mathrm{mult}_Q(D) = \mathrm{mult}_{P_i}(F) - k + \mathrm{mult}_Q(\widetilde{F}).$$

But $\mathrm{mult}_Q(\widetilde{F}) \leq \mathrm{mult}_{P_i}(F)$, thus we finally obtain the statement (2.1). ∎

3. The points fattening effect on $\mathbb{S}_1$

In this section we present some results concerning the points fattening effect on $\mathbb{S}_1$ taken as a del Pezzo surface. Let us recall, that $\mathbb{S}_1$ arises as the blow-up of the projective plane in a fixed point. To keep the notation consistent we denote this point by P_1 and by E_1 we denote the exceptional divisor of this blow-up. Basic questions when studying the problem of points fattening on an arbitrary variety are: What is the minimal growth of the initial sequence and how can the sets on which this minimal growth happens be characterized geometrically. By the minimal growth (or minimal jump) we understand the minimal difference between the consecutive numbers of the initial sequence. For the surface $\mathbb{S}_1$ this minimal growth is 0 and moreover there is possible to get more than one zero jump. We begin with the characterization of sets with the maximal number of such zero jumps, namely 5.

Theorem 3.1
Let $Z \subset \mathbb{S}_1$ be a finite set of points. Then the following conditions are equivalent

i) $Z = \{Q\} \subset E_1$,

ii) $\alpha(Z) = \alpha(2Z) = \alpha(3Z) = \alpha(4Z) = \alpha(5Z) = 1$.

Proof. The implication from *i)* to *ii)* is obvious. It is enough to consider the nonreduced curve $F = 3L \subset \mathbb{P}^2$ for some line passing through the point P_1. Indeed, it gives rise to

$$f_a^*F = f^*F - E_1 = 3\widetilde{L} + 2E_1$$

in $\mathbb{S}_1$, which vanishes to order 5 along $Q \in \widetilde{L} \cap E_1$.

In order to prove the reverse implication let $Z = \{Q_1, \ldots, Q_s\}$ and we assume that $D \in |3H - E_1|$ is a divisor satisfying $\mathrm{mult}_{Q_i}(D) \geq 5$ for all points $Q_i \in Z$. First we will prove that $Z \subset E_1$. Suppose to the contrary, that there exists $Q \in Z$ such that $Q \in \mathbb{S}_1 \setminus E_1$ and $\mathrm{mult}_Q(D) \geq 5$. Let $F = f(D)$. Then $\deg(F) = 3$, but $\mathrm{mult}_{f(Q)}(F) \geq 5$. We obtained a cubic curve with a quintuple point, what contradicts with the statement (2) of Lemma 2.3 with $k = 1$. That means $Z \subset E_1$.

Now let us consider possible types of cubic curves in the projective plane and their adapted transforms. The curve F has to pass through the point P_1 (because $F = f(D)$) and it should have the highest possible multiplicity in this point (in order to get the highest possible multiplicities along the exceptional divisor E_1). We have the following types of cubic curves on $\mathbb{P}^2$:

a) irreducible cubic (possibly singular), or

b) a union of an irreducible conic and a line, or

c) a union of three lines (possibly not distinct).

In case a) the divisor f_a^*F on $\mathbb{S}_1$ has points of multiplicity at most two. In case b) the highest possible multiplicity of a point on E_1 is three, this happens in the case when the line is tangent to the conic at point P_1.

Let us pass to the case *c*). To get possibly high multiplicities of points on E_1 there is only one condition to satisfy for three component lines: they have to pass through the point P_1 as many times as possible, but at least once. This forces specific arrangements of these lines. We know, that the adapted transform of a curve F consisting of some triple line L has quintuple point. Except for this one arrangement we may consider two more situations when P_1 has the maximal multiplicity (namely 3), i.e. three distinct lines or one single and one double line passing through P_1, but any of them give the quintuple points (see at the distinguished points $Q_i \in E_1$ in the Figures 5, 6 and 7).

■

Remark 3.2
In fact there does not exist any other set beside of sets from Theorem 3.1 satisfying the condition

$$\alpha(mZ) = \alpha((m+1)Z) = \alpha((m+2)Z) = \alpha((m+3)Z).$$

for some $m \geq 1$ on $\mathbb{S}_1$. Moreover for any integer m we have

$$\alpha(mZ) < \alpha((m+5)Z).$$

(see details in [8], Chapter 6).

The estimation given in Remark 3.2 let us to establish explicit formula for the initial sequence in this case and find its Waldschmidt constant.

Lemma 3.3
If $Z = \{Q\}$ and $Q \in E_1$, then $\alpha(mZ) = \lceil \frac{m}{5} \rceil$ and $\widehat{\alpha}(Z) = \frac{1}{5}$.

Proof. Let us first notice, that the divisor $F = 3kL$ for the line L passing through the P_1 and corresponding to the point $Q \in E_1$, gives rise to $D = 3k\widetilde{L} + 2kE_1 \in |3kH - kE_1|$ on the blow up $\mathbb{S}_1$ and $\text{mult}_Q(D) = 5k$ for any $Q \in Z$. Hence $\alpha(5kZ) \leq k$ for any positive integer k.

For $k = 1$ we then obtain $\alpha(5Z) \leq 1$, what means that

$$\alpha(Z) = \ldots = \alpha(5Z) = 1.$$

Moreover from Remark 3.2 we conclude

$$\alpha(6Z) \geq 2. \tag{3.1}$$

On the other hand for $k = 2$ we have

$$\alpha(10Z) \leq 2. \tag{3.2}$$

From (3.1) and (3.2) we then obtain $\alpha(6Z) = \ldots = \alpha(10Z) = 2$.

Using the same argumentation for the next k we finally conclude, that the initial sequence in this case is $\alpha(mZ) = \lceil \frac{m}{5} \rceil$. We pass to the proof of the second statement.

Let us notice, that we have the following obvious sequence of inequalities

$$\frac{m}{5} \leq \left\lceil \frac{m}{5} \right\rceil \leq \frac{m}{5} + 1.$$

Dividing all terms by m we obtain

$$\frac{1}{5} \leq \frac{\lceil \frac{m}{5} \rceil}{m} \leq \frac{m+5}{5m}.$$

Obviously $\lim_{m\to\infty} \frac{m+5}{5m} = \frac{1}{5}$, what implies

$$\widehat{\alpha}(Z) = \lim_{m\to\infty} \frac{\lceil \frac{m}{5} \rceil}{m} = \frac{1}{5}.$$

∎

The consequence of Lemma 3.3 and Remark 3.2 is the following result.

Corollary 3.4
Let $\alpha(mZ)$ be an initial sequence for some finite set $Z \subset \mathbb{S}_1$. Then $\widehat{\alpha}(Z) \geq \frac{1}{5}$.

Proof. The idea is to compare the sequence $\alpha(mZ)$ with the sequence $a_m = \lceil \frac{m}{5} \rceil$. To this end observe that

$$\alpha(mZ) \geq a_m \tag{3.3}$$

for any integer m. Then also

$$\frac{\alpha(mZ)}{m} \geq \frac{a_m}{m} \geq \frac{1}{5}$$

for any integer m. ∎

On $\mathbb{S}_1$, there also exist infinitely many sets satisfying the weaker condition, namely

$$\alpha(mZ) = \ldots = \alpha((m+3)Z),$$

and these sets are not necessarily the same as in Theorem 3.1. Let us recall, that in the case of the line bundle $2H - E_1$ for $\mathbb{S}_1$ taken as a Hirzebruch surface this condition also would be never fulfilled.

Theorem 3.5
Let $Z \subset \mathbb{S}_1$ be a finite set of points and let m be a positive integer. Then the following conditions are equivalent

i) $\alpha(mZ) = \ldots = \alpha((m+3)Z)$

ii) $Z = \{Q\} \subset E_1$ *or* $Z = \{Q_1, Q_2\} \subset E_1$, *where* $Q_1 \neq Q_2$.

Proof. The sets in *ii*) satisfy the condition

$$\alpha(mZ) = \ldots = \alpha((m+3)Z),$$

for example with $m = 1$ and $m = 4$ respectively. We will prove the opposite implication.

Firstly, by Lemma 2.3 we conclude, that $Z \subset E_1$. Suppose now, that $Z = \{Q_1, \ldots, Q_t\}$ is a set such that $\alpha(mZ) = \ldots = \alpha((m+3)Z) = k$ for some integers k and t and let $D \in |3kH - kE_1|$ be a divisor such that $\text{mult}_{Q_i}(D) \geq m+3$ for any point $Q_i \in Z$. Let us denote by $F = f(D)$, with $\deg(F) = 3k$.

We have the following estimates. Since F is of degree $3k$, its multiplicity at P_1 is at most $3k$. Hence the multiplicity of E_1 in D is at most $2k$. This contributes to the multiplicity of D at every point $Q_1, \ldots, Q_t$. The remaining multiplicity at these points must come from branches of F passing through P_1 at directions corresponding to $Q_1, \ldots, Q_t$. We have

$$t(m+3) \leq \sum_{i+1}^{t} mult_{Q_i} D \leq 3k + 2kt. \tag{3.4}$$

On the other hand, since $\alpha(mZ) = k$, it must be

$$3(k-1) + 2(k-1)t < t \cdot m, \tag{3.5}$$

since otherwise one could find $3(k-1)$ lines through P_1, which pull-back to $\mathbb{S}_1$ would show $\alpha(mZ) \leq k-1$ contradicting the assumption. Combining (3.4) and (3.5) we get

$$3k - 3 + 2kt - 2t + 3t < 3k + 2kt$$

and thus $t < 3$.

■

Remark 3.6
In the case of set $Z = \{Q\} \subset E_1$ in fact we have even stronger condition, than i) (compare to Theorem 3.1).

Remark 3.7
The smallest integer m, where

$$\alpha(mZ) = \ldots = \alpha((m+3)Z)$$

holds in Theorem 3.5 is 1, if $Z = \{Q\}$ and 4, when $Z = \{Q_1, Q_2\}$.

In Figure 3 we present the set $Z = \{Q_1, Q_2\}$ described in the previous theorem with the curve giving the beginning values of the initial sequence, i.e.

$$\alpha(Z) = \alpha(2Z) = \alpha(3Z) = 1$$

and

$$\alpha(4Z) = \alpha(5Z) = \alpha(6Z) = \alpha(7Z) = 2.$$

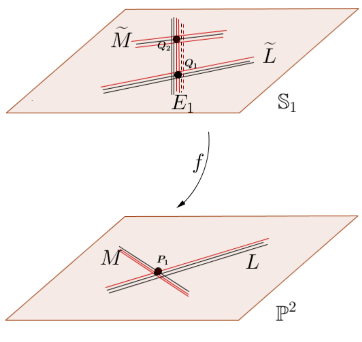

Figure 3: Two distinct points on E_1

To see the multiplicity 5 at the $Z = \{Q\} \subset E_1$, look at Figure 6 (more precisely, at the distinguished point Q).
Generally we can prove the following property (see more in [8], Chapter 2 and 6).

Lemma 3.8
If $Z \subset \mathbb{S}_1$ satisfies the condition

$$\alpha(mZ) = \ldots = \alpha((m+t)Z)$$

for some positive integers m and $t \geq 3$, then $Z \subset E_1$.

Proof. It is immediate from Lemma 2.3. ∎

We may also consider the sets with three initial values equal 1. All such possibilities are described in the following theorem. The proof is analogous to the proof of Theorem 3.1 and it is based on the review of well known types of the plane cubic curves (it was just listed in the proof of Theorem 3.1). We then omit details this time and present only the statement of theorem.

Theorem 3.9
A set $Z \subset \mathbb{S}_1$ satisfies the condition

$$\alpha(Z) = \alpha(2Z) = \alpha(3Z) = 1,$$

if and only if Z is one of the following sets:

a) $Z = \{Q\}$,

b) $Z \subset \{Q_1, Q_2, Q_3\} \subset E_1$,

c) $Z \subset \widetilde{L}$, *where L is an arbitrary line passing through the point P_1,*

d) $Z \subset \{Q_1, Q_2\}$, *where Q_1 is an arbitrary point on E_1 and Q_2 is an arbitrary point outside of E_1.*

We present all of these sets with curves realising the condition

$$\alpha(Z) = \alpha(2Z) = \alpha(3Z) = 1$$

in Figures 4, 5, 6 and 7.

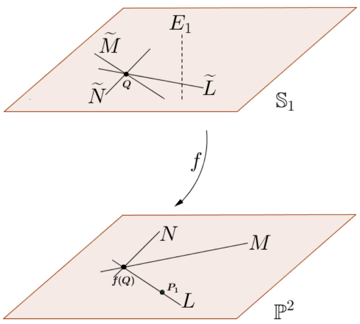

Figure 4: One triple point

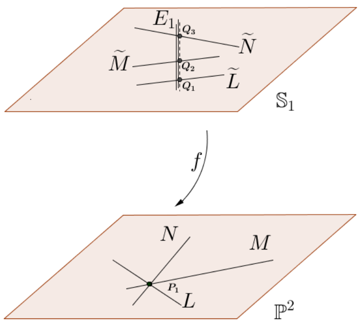

Figure 5: $Z \subset E_1$

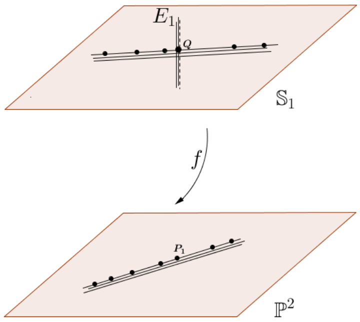

Figure 6: Collinear points

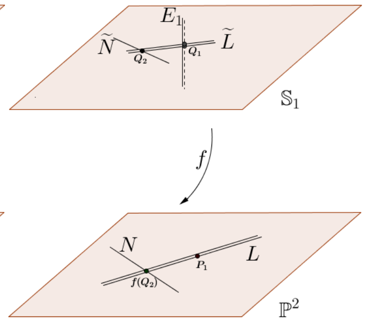

Figure 7: The "mixed" set

It is worth to mention here, that although in the Figure 4 we present point $Q \notin E_1$, the case a) of Theorem 3.9 includes also the situation, where $Q \in E_1$. From Theorem 3.1 we know, that if $Q \in E_1$ the equality $\alpha(3Z) = 1$ also holds. Of course, three $0-$jumps may appear later, not necessary on the beginning of the initial sequence. It is illustrated in the following example.

Example 3.10
If $Z = \{Q_1, Q_2\}$ and $Z \cap E_1 = \emptyset$, then $\alpha(Z) = \alpha(2Z) = 1$ and $\alpha(3Z) = \alpha(4Z) = \alpha(5Z) = 2$ (see Figure 8).

Although we are mainly interested to make characterization of sets with maximal number of $0-$jumps, sometimes we meet some sets with less number of them, but having interesting behaviour from some points of view. We conclude with an example of such a set.

Example 3.11
Let $Z = \{Q_1, Q_2, R_1, R_2\}$, where $f(R_1)$ and $f(R_2)$ are arbitrary points distinct of P_1 and $Q_1, Q_2 \in E_1$ and moreover Q_i lies on the proper transform of the line joining $f(R_i)$ with P_1 for $i \in \{1, 2\}$ (see Figure 9).

Although the constant initial sequence here is short (only two numbers equal 1, instead of possible five) this example is very interesting because of symmetry between the points. Let us notice, that every two of points: Q_1 and Q_2, R_1 and R_2, Q_1 and R_1, Q_2 and R_2 always lie on a common line, being a component of a divisor realising the equality

$$\alpha(Z) = \alpha(2Z) = 1.$$

Moreover points Q_i and R_i are strongly related for fixed $i \in \{1, 2\}$. Since the point $f(R_i)$ determines the direction of a line passing through P_1, then in fact it determines the position of the point Q_i on E_1.

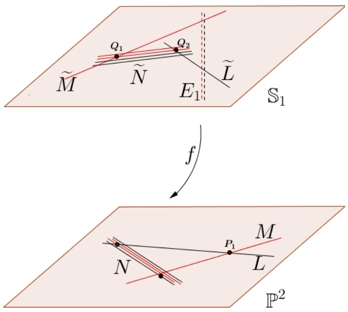

Figure 8: Example 3.10

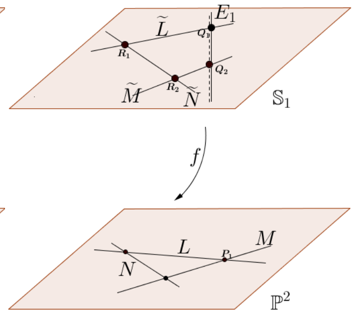

Figure 9: Example 3.11

Much more interesting results concerning the points fattening effect on $\mathbb{S}_1$ and also on the remaining del Pezzo surfaces $\mathbb{S}_r$ reader can find in [8].

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[1] *Instytut Matematyki*
Uniwersytet Pedagogiczny w Krakowie
30-084 Kraków, ul. Podchorążych 2
E-mail: lampa.baczynska@wp.pl

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