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*Karol Gryszka*[1]

# Hausdorff Dimension is onto

**Abstract.** The aim of this article is to solve the following problem: given any positive real number $h$ construct a set $X$ such that its Hausdorff dimension is equal to $h$. We obtain the result via generalized Cantor sets in arbitrary dimension. We also discuss the following question: how many topologically different sets can have common Hausdorff dimension?

## 1. Introduction

In the theory of fractals one of fundamental notions is the Hausdorff dimension. Such a dimension measure is in some sense the complexity of a set, although one should not think of it as a very precise way of the description of fractal sets. An interesting phenomenon is the number itself need not be an integer; actually in many cases it is an irrational number.

The question that the author posed is if any number could be obtained as a Hausdorff dimension of some set. If yes, then how many such sets one can construct? In this article we present our approach to this problem along with some auxiliary comments. The main results are in Sections 3 and 4, where we essentially prove that there are uncountably many topologically different sets with common Hausdorff dimension. Section 2 introduces notation and basic definitions along with some examples and remarks concerning classical fractal sets, which become important once again in Section 3.

## 2. Preliminaries

Throughout the article we denote the unit interval $[0, 1]$ by $I$. The cardinality of a set $A$ is denoted by $\#A$.

The main definition in our consideration is the Hausdorff dimension. Let us recall it along with other necessary notions (see [3] and [6]). Let $X$ be any subset of $\mathbb{R}^n$ and let $\{U_j\}_{j\in J}$ be a countable or finite family of sets of diameter at most $\delta$ that cover $X$. In that case we say that $\{U_j\}_{j\in J}$ is a $\delta$-cover of $X$.

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Definition 2.1
For any $\delta > 0$ and define

$$H^s_\delta(X) = \inf\left\{\sum_{j\in J}(\mathrm{diam}(U_j))^s \mid \{U_j\}_{j\in J} \text{ is a } \delta - \text{cover of } X\right\}.$$

Then, as $\delta$ decreases, $H^s_\delta(X)$ increases. We denote

$$H^s(X) = \lim_{\delta\to 0} H^s_\delta.$$

The value $H^s(X)$ is called the *$s$-dimensional Hausdorff measure of $X$*.

Depending on $s$ we usually have $H^s(X) = 0$ or $H^s(X) = \infty$. However, the graph $s \mapsto H^s(X)$ jumps, as $s$ rises, from $\infty$ to 0 at a unique point $s$, called the Hausdorff dimension.

Definition 2.2
The *Hausdorff dimension* $\dim_H(X)$ of $X$ is defined by

$$\dim_H(X) = \inf\{s > 0 \mid H^s(X) = 0\} = \sup\{s > 0 \mid H^s(X) = \infty\}.$$

The Hausdorff dimension is usually difficult to evaluate, however in some special cases like $X$ being an attractor of some IFS we can calculate it relatively simply.

Following Chapter 2 in [4] we recall basic definitions and properties of IFS. We say that $f \in Con_c(X)$ if $f : X \to X$ is a contraction mapping with Lipschitz constant $c < 1$, that is $d(f(x), f(y)) < c \cdot d(x, y)$ for any $x \neq y$. From now on we will refer to $c$ as contraction constant.

Let $X$ be a complete metric space.

Definition 2.3
Any finite set $\mathcal{F}$ of contractions of $X$ is called an *iterated function system* (abbreviated *IFS*).

From now on whenever we are given some IFS, we also have a complete metric space $X$ (in our case it is $\mathbb{R}^n$ for some $n > 0$), the number of functions $k$ and functions $f_i \in Con_{c_i}(X)$ for $i = 1, \ldots, k$. We also write $F = f_1 \cup \ldots \cup f_k$, i.e. if $B \subset X$, then $F(B) = f_1(B) \cup \ldots \cup f_k(B)$. Then,

$$F^n(B) = f_1(F^{n-1}(B)) \cup \ldots \cup f_k(F^{n-1}(B))$$

for any $n > 1$.

Banach fixed-point theorem plays significant role in the theory. This is due to the following theorem.

Theorem 2.4 (Hutchinson, [5])
*Let $\mathcal{F}$ be an IFS. Then there exists exactly one non-empty compact set $A \subset X$ such that $F(A) = A$.*

The set $A$ in the above theorem is called *an attractor* of $\mathcal{F}$. By the Banach fixed-point theorem, any non-empty compact set $B$, called *the initial set* or *the starting set*, converges in the Hausdorff metric $d_H$ to the set $A$, that is

$$\lim_{n\to+\infty} d_H(F^n(B), A) = 0.$$

Before we state a theorem that shows how to easily compute the Hausdorff dimension, recall that $\mathcal{F}$ satisfy the *open set condition* if there exists an open and bounded set $V \neq \emptyset$ such that $F(V) \subset V$ and $f_i(V), i = 1, \dots, k$ are disjoint. Recall also that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is called *similarity* if it is a composition of rotations, translations and uniform scaling along all coordinates. In that case if $f$ is also a contraction, then the contraction rate $c$ is the scaling factor, that is $d(f(x), f(y)) = c \cdot d(x, y)$ for any $x$ and $y \in \mathbb{R}^n$.

Theorem 2.5 (Moran, Theorem II in [8])
*Suppose that $\mathcal{F}$ satisfies the open set condition and each $f_i \in \mathcal{F}$ is a similarity. If $A$ is an attractor such that $F(A) = A$, then $\dim_H A = s$, where $s$ is a unique solution of the equation*

$$\sum_{i=1}^{k} c_i^s = 1.$$

We now present two important examples of fractal sets along with their generalizations. The latter are obtained by changing the key parameter defining the set. These examples will eventually give us some topologically different sets with equal Hausdorff dimension.

Example 2.6
The Cantor set $C \subset \mathbb{R}$ is the attractor of the IFS with

$$\begin{aligned} f_1(x) &= \frac{1}{3}x, \\ f_2(x) &= \frac{1}{3}x + \frac{2}{3}, \end{aligned}$$

and the starting set $I$. The usual construction is as follows: let $C_0 = I$ and for $n > 0$ define inductively the set $C_n$ by diving each component segment of $C_{n-1}$ into three equal segments and removing the middle one (see Figure 1). Then $C = \bigcap_{n\geq 0} C_n$ defines the classical Cantor set. Moreover, $C_n = F^n(I)$ and $\{f_1, f_2\}$ satisfies the open set condition (take for example $V = (0, 1)$)).

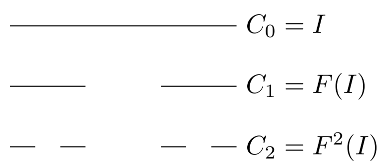


Figure 1: Construction of the Cantor set.

By Moran's Theorem the Hausdorff dimension $s$ of $C$ is the solution to the following equation:

$$\frac{1}{3^s}+\frac{1}{3^s}=1,$$

therefore

$$\dim_H C = s = \frac{\log 2}{\log 3}.$$

We now modify the construction of the Cantor set - this will lead us to a wider range of fractional dimension.

For a fixed $a \in (0,1)$ we consider the IFS with the following functions:

$$f_1^a(x) = \frac{1-a}{2}x,$$
$$f_2^a(x) = \frac{1-a}{2}x + \frac{1+a}{2},$$

and the initial set $I$. Let us denote the unique attractor of that IFS by $C^a$. Using Moran's Theorem we can evaluate the Hausdorff dimension of $C^a$ similarly as for the classical Cantor set to obtain

$$\dim_H C^a = \frac{\log 2}{\log 2 - \log(1-a)}.$$

We can now check that

$$\lim_{a\to 0} \dim_H C^a = 1,$$
$$\lim_{a\to 1} \dim_H C^a = 0,$$

and the function

$$(0,1) \ni a \mapsto \dim_H C^a = \frac{\log 2}{\log 2 - \log(1-a)} \in (0,1)$$

is continuous, therefore it is onto $(0,1)$.

Generalized Cantor set plays important role in the next section, where it is used to build a set with arbitrary Hausdorff dimension.

Example 2.7
The Koch curve $K \subset \mathbb{R}^2$ is constructed in the following way: embed the interval $I$ in $\mathbb{R}^2$ and divide it into three segments, draw an equilateral triangle with the middle segment of the division as its base, remove the base of the triangle. The process in then repeated for each subsequent segment ad infinitum. The initial steps of the construction are presented in Figure 2.

More formally, the Koch curve is the attractor of the IFS consisting of 4 contractions with the common contraction constant $c = \frac{1}{3}$ (we skip the formulas for the contractions). The open set condition holds, taking for instance $V$ as an interior of an isosceles triangle with base 1.

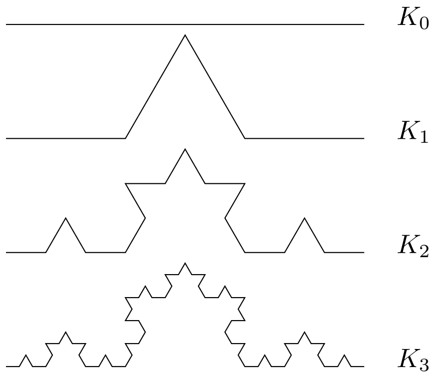


Figure 2: Construction of the Koch curve.

By Moran's Theorem the Hausdorff dimension is

$$\dim_H K = \frac{\log 4}{\log 3}.$$

The generalization of that curve is based on varying angle between the two non-horizontal segments in $K_1$ (notation according to Figure 2). The angle however cannot be chosen at will - by [2] there are certain values of it that generate self-intersections of the set (and thus the open set condition cannot hold). This is not the case for the angle $\alpha$ in the range $\left(\frac{\pi}{3}, \pi\right)$ to which we limit our further consideration. Then, if the starting set $K_0$ is $I$ and $a$ denotes the length of one segment in Figure 3, then

$$a = \frac{1}{2 + 2\sin\frac{\alpha}{2}}.$$

Clearly, $a$ is also the common contraction constant of 4 functions forming the IFS with the set $K_\alpha$ as its attractor.

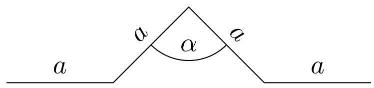


Figure 3: The generalization of the Koch curve with angle parameter $\alpha$. The picture shows the second step of the construction.

Using the same argument as above we conclude that

$$\dim_H K_\alpha = \frac{\log 4}{\log\left(2 + 2\sin\frac{\alpha}{2}\right)}.$$

It is now easy to verify (in similar to Cantor sets fashion) that

$$\left(\frac{\pi}{3}, \pi\right) \ni \alpha \mapsto \dim_H K_\alpha = \frac{\log 4}{\log\left(2 + 2\sin\frac{\alpha}{2}\right)} \in (1, \log_3 4)$$

is continuous and onto.

Example 2.8
We will use the idea from the Koch curve construction to build a subset of $\mathbb{R}^3$ with Koch-like structure and with varying Hausdorff dimension. We start with the square $I^2$ embedded in $\mathbb{R}^3$. Then we replace it with the surface presented in Figure 4. Then each of the 12 new squares is subsequently replaced by the scaled copy of the surface. Clearly, such an operation describes an IFS consisting of 12 contractions with the same contraction constant. The open set condition holds, taking for instance $V$ as an interior of a suitable cube with base 1.

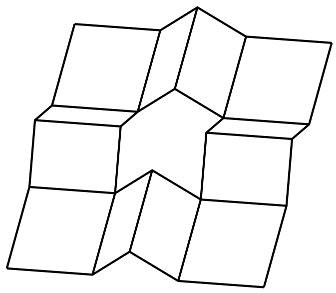

Figure 4: Koch-like set $L$ in third dimension. The picture shows the second step of the construction. The surface is rotated in space to show all the details.

The Hausdorff dimension of that set satisfies the equation

$$12 \cdot \left(\frac{1}{3}\right)^{\dim_H L} = 1,$$

according to Moran's Theorem. From this it follows that

$$\dim_H L = \frac{\log 12}{\log 3}.$$

We can now generalize the above construction as in Example 2.7 by changing the angle between two middle squares. This can lead to a set with Hausdorff dimension having any value between 2 and $\log_3 12$.

During the research and preparing this paper the author was surprised that there is no direct generalization of the Koch Curve to third dimension. One of ideas could be the tetrahedron whose faces are divided in 4 equal isosceles triangles, then the middle one is replaced by a scaled copy of the initial tetrahedron. This process could now continue for each of 24 smaller faces of the new polyhedron. The actual problem of that is the limit of such a process is just a cube.

One of promising approaches in third dimension was provided in [1], the other in [7] (the former is based on cutting tetrahedron, while the latter is based on building cubes around other cubes). As for author's knowledge there is no general approach to any dimension.

## 3. Surjection property

In this section we prove that each positive real number is a Hausdorff dimenions of some set. Since $\dim_H I^n = n$, we skip sets with the integer Hausdorff dimension.

THEOREM 3.1
*Given any positive real number $h$ there exists a number $n > 0$ and a set $X \subset \mathbb{R}^n$ such that* $\dim_H X = h$*.*

*Proof.* We use the idea from Example 2.6 and construct Cantor-like set in a specific dimension. Namely, fix $h > 0$ and set

$$\begin{aligned} n &:= \lceil h \rceil, \\ a &:= 1 - 2^{1-\frac{\lceil h \rceil}{h}}. \end{aligned}$$

Let $C_0 = I^n$ and consider two contractions defined on $I$:

$$\begin{aligned} f_1(x) &= \frac{1-a}{2}x, \\ f_2(x) &= \frac{1-a}{2}x + \frac{1+a}{2}. \end{aligned}$$

Then $f_1$ and $f_2$ build the IFS with the starting set $I$. Take $D_k = F^k(I)$ and

$$C_k = (D_k)^n = \underbrace{D_k \times \ldots \times D_k}_{\times n}.$$

Finally, define $C^h = \bigcap_{k \geq 0} C_k$. Then $\dim_H C^h = h$. Indeed, by the construction of the IFS and the sets $D_k$ each $C_{k+1}$ is a scaled copy of $2^n$ sets $C_k$ with scaling ratio $\frac{1-a}{2}$. This allows to define the IFS consisting of $2^n$ contractions with mutual contraction constant $c = \frac{1-a}{2}$. Then by Moran's Theorem, the Hausdorff dimension of $C^h$ is the unique solution to the equation

$$2^n \cdot \left(\frac{1-a}{2}\right)^{\dim_H C^h} = 1.$$

Then, using the definition of $n$ and $a$ (and some calculations from Example 2.6),

$$\begin{aligned} \dim_H C^h &= \frac{\log 2^n}{\log 2 - \log(1-a)} \\ &= \frac{n \log 2}{\log 2 - \log\left(1 - 1 + 2^{1-\frac{\lceil h \rceil}{h}}\right)} \\ &= \frac{\lceil h \rceil \log 2}{\left(1 - 1 + \frac{\lceil h \rceil}{h}\right) \log 2} \\ &= h. \end{aligned}$$

The proof is completed. ■

## 4. Cardinality

From Section 2 it follows that for a certain range there are at least two non-homeomorphic sets having equal Hausdorff dimension. In this section we prove that there are uncountably many such sets. We begin with the following Lemma.

Lemma 4.1
*If $X$ is a countable set, then* $\dim_H X = 0$.

*Proof.* Pick any $s > 0$. Our goal is to show that $H^s(X) = 0$ or equivalently, to show that for any $\varepsilon > 0$ there is $\delta > 0$ such that $H^s_{\delta'}(X) < \varepsilon$ for any $\delta' < \delta$.

Pick any $\varepsilon > 0$ and take $\delta < \left(\frac{\varepsilon}{2}\right)^{1/s}$. Arrange $X$ into the sequence $(a_j)_{j\in\mathbb{N}}$ and take opens sets $U_j$ so that $a_j \in U_j$ and $(\mathrm{diam}(U_j))^s < \frac{\varepsilon}{2^{j+1}}$. Then $\{U_j\}_{j\in\mathbb{N}}$ is a $\delta$-cover of $X$ and

$$H^s_\delta(X) \le \sum_{j\in\mathbb{N}} (\mathrm{diam}(U_j))^s \le \sum_{j\in\mathbb{N}} \frac{\varepsilon}{2^{j+1}} = \varepsilon.$$

Having the above and repeating the argument we conclude that if $\delta' < \delta$, then also $H^s_{\delta'}(X) < \varepsilon$. Consequently,

$$\lim_{\delta\to 0} H^s_\delta(X) = 0.$$

Finally, since $s$ was arbitrary, the Hausdorff dimension of $X$ equals 0. ■

Theorem 4.2
*For each $h > 0$ there are uncountably many pairwise non-homeomorphic sets having Hausdorff dimentions equal to $h$.*

*Proof.* We divide the construction into several steps.
**Step 1.** Let $C^h$ be the set as in the proof of Theorem 3.1 with $\dim_H C^h = h$. If $\lceil h \rceil = 1$, embed the set $C^h$ in $\mathbb{R}^2$ using the mapping $x \mapsto (x, -10)$ and define $n = 2$. Otherwise, let $n = \lceil h \rceil$ and use the embedding $x \mapsto (x, -10, \ldots, -10) \in \mathbb{R}^n$. In either case we abuse the notation and denote the embedded set with the same symbol $C^h$.
**Step 2.** Consider the following set $L \subset \mathbb{R}^2$. It is built from a half-line $[0, +\infty) \times \{0\}$ with loops disjoint from $\mathbb{R} \times \{0\}$ attached to points $x_n := (n, 0)$ and they are contained in the half-plane $y \ge -3$. To each $x_n$ we attach exactly $4n + 1$ disjoint loops. See Figure 5 for an overview.

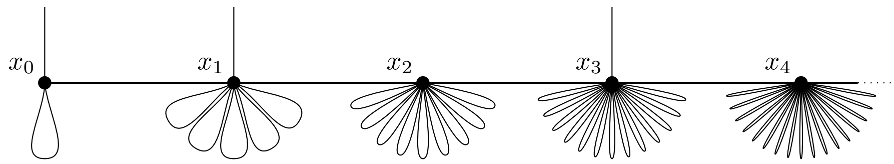


Figure 5: A sketch of initial part of the set $L_\sigma$ for $\sigma = (1, 1, 0, 1, 0, \ldots)$.

**Step 3.** Take any $\sigma \in \{0,1\}^{\mathbb{N}}$ and associate with it extra segments attached to the set $L$ by the following rule: if $\sigma(n) = 1$, then attach to the point $x_n$ one segment that has only one common point with $L$ - the point $x_n$, otherwise attach nothing. See Figure 5 for a sketch. Let $L_\sigma$ be the set $L$ with the attached segments according to the above rule and let $x_n^\sigma = x_n$ for all $n \in \mathbb{N}$.
**Step 4.** Embed $L_\sigma$ in $\mathbb{R}^n$ by adding zeros to remaining $n-2$ coordinates:

$$f : \mathbb{R}^2 \ni (x, y) \mapsto (x, y, \underbrace{0, \ldots, 0}_{\times(n-2)}) \in \mathbb{R}^n$$

and denote $K_\sigma := f(L_\sigma)$. Take $D_\sigma^h = K_\sigma \cup C^h$. The choice of $-10$ in Step 2. is to ensure $K_\sigma$ and $C^h$ are disjoint sets.

From the above construction it follows that if $\sigma \neq \tau$, then $D_\sigma^h$ is not homeomorphic to $D_\tau^h$. Indeed, since any two Cantor sets are totally disconnected and are homeomorphic to one another, it is enough to show that property for the sets $L_\sigma$ from Step 3. Take any $\sigma$ and $\tau$ and consider the sequences $(x_n^\sigma)_{n\in\mathbb{N}}$ and $(x_n^\tau)_{n\in\mathbb{N}}$.

Assume that $L_\sigma$ is homeomorphic to $L_\tau$ via homeomorphism $T$. Note that $L_\sigma \setminus \{x_0^\sigma\}$ has exactly $2 + \sigma(0)$ components, therefore the set $T(L_\sigma \setminus \{x_0^\sigma\})$ must have as many. This can happen only when $T(x_0^\sigma) = x_0^\tau$. We now proceed step-by-step to show that $T(x_n^\sigma) = x_n^\tau$ for each $n \in \mathbb{N}$: if $k > 0$ and $L_\sigma \setminus \{x_k^\sigma\}$ has $(4 \cdot k + 1) + 2 + \sigma(n)$ components, so does the set $T(L_\sigma \setminus \{x_k^\sigma\})$ and that is the case only when $T(x_k^\sigma) = x_k^\tau$

The function $T$ is a homeomorphism, therefore each loop attached to $x_n^\sigma$ is mapped to some loop attached to $x_n^\tau$. Similarly, if $[x_n^\sigma, x_{n+1}^\sigma]$ is the segment joining $x_n^\sigma$ and $x_{n+1}^\sigma$, then its image joins points $x_n^\tau$ and $x_{n+1}^\tau$ for all $n \in \mathbb{N}$. Finally, each segment attached to $x_n^\sigma$ in Step 3. is mapped to a segment attached to $x_n^\tau$.

From the above it follows that $\sigma(n) = \tau(n)$ for any $n \in \mathbb{N}$, and consequently $\sigma = \tau$.
**Step 5.** We now slightly modify Step 2 and Step 3: we change the loops (via some homeomorphism) in such a way that if $\ell$ is any loop in $L_\sigma$, then $\ell \cap \mathbb{Q}^2$ is dense in $\ell$. Thus, $L_\sigma \cap \mathbb{Q}^2$ is dense in $L_\sigma$.

We conclude that if $L_\sigma \cap \mathbb{Q}^2$ is homeomorphic to $L_\tau \cap \mathbb{Q}^2$, then $L_\sigma$ is such to $L_\tau$. The latter can happen only if $\sigma = \tau$.

We can now proceed as in Step 4 to complete the construction by replacing $L_\sigma$ with $L_\sigma \cap \mathbb{Q}^2$; however we abuse the notation and set $L_\sigma$ to be $L_\sigma \cap \mathbb{Q}^2$.

We conclude that the sets $L_\sigma$ are pairwise non-homeomorphic to one another. Clearly, since $\#\{0,1\}^{\mathbb{N}} > \aleph_0$, there are uncountably many such sets.

It is remaining to check that $\dim_H D_\sigma^h = h$. Indeed, by Lemma 4.1 we have $\dim_H K_\sigma = 0$ and therefore

$$\dim_H D_\sigma^h = \max\{\dim_H K_\sigma, \dim_H C^h\} = \max\{0, h\} = h$$

for any $\sigma \in \{0,1\}^{\mathbb{N}}$. ■

The above proof takes advantage of the cardinality of both rational numbers and their power set. Since the cardinality of the latter is strictly smaller than the cardinality of all subsets of $\mathbb{R}^n$ it is interesting to know if one can construct $\#\mathcal{P}(\mathbb{R})$ many pairwise non-homeomorphic sets satisfying the property described in Theorem 4.2. We leave that as an open problem.

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[1] *Instytut Matematyki*
*Uniwersytet Pedagogiczny w Krakowie*
*ul. Podchorążych 2, 30-084 Kraków*
*E-mail:* karol.gryszka@up.krakow.pl

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