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The role of visualisation in the process of generalisation*

Abstract. In my paper, I discuss the results of individual observations of pupils aged 13-14. These children undertook an attempt to solve problems which were designed in order to bring about an inductive type of generalizing. The main aim of the study was to classify typical methods of proceeding, which represent the pupils' ways of reasoning and to determine what influence a particular method of proceeding may have on the final outcome of their work. These results indicate the general pupils' strategies of generalizing. Presumably, visual thinking produces a positive effect on the pupils' process of generalizing and abilities to describe regularity with the aid of a letter symbol.

1. Introduction

In modern mathematics education, mastering the approach is valued more than mastering the knowledge. This is why great importance is attached to developing the mathematical activity of the students. The preferred methods allow students to discover or create mathematics. According to J. Filip and T. Rams: *Children should learn to communicate, articulate, and describe their ideas and doubts, be allowed attempts and mistakes, cooperate with each other.* They need to learn to listen to others, and, most importantly, learn on their own, "find their own approach to mathematics" (Filip, Rams, 2000, p. 9). In order to shape this approach, mathematical activities and behaviour should be nourished. A. Z. Krygowska outlined a number of such activities, including the perception and use of analogy, creating schemes, assimilation and processing of information, deduction, reducing problems into others, interpretation and use of definitions, creating and using algorithms (Krygowska, 1986), as well as generalizing (Krygowska, 1981).

The need of introducing elements of generalisation-based reasoning can be noticed in the recent years in teaching centres both in Poland as well as worldwide.

*Rola wizualizacji w procesie uogólniania

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This is influenced by the international PISA research (*Programme for International Student Assessment*), the aim of which is to assess the knowledge, skills, and stance of 15 year old students in the context of the three concepts which were deemed most important in the development of teenagers. Emphasis is placed on *reading and reasoning in humanities, mathematics* as well as *reasoning in natural sciences.* In the context of mathematics, the skills taken into consideration include, among others, the ability of abstract thinking, analysis, and generalisation. In the context of the development of such activities, the need for the didactic activities of the teacher are also worth taking into consideration. The aim of the activities is to *develop the habit of observation, experimentation, seeking and attaining knowledge on one's own; this requires, on all levels, teaching the development of perception and use of analogies (similarities and differences), empirical inference, use of recurrent reasoning, and* ***inductive generalisation*** (Legutko, Stańdo, 2008, p. 32; own emphasis). The aforementioned mathematical activities can be developed on the basis of geometry and arithmetic. In the recent years, a significant amount of mathematics education literature is dedicated to the process of generalisation (Presmeg, 1999; Ciosek, 1995; Zaksis, Liljedahl, 2002; Legutko, 2010; Flores, 2012; Hitt, 2012; Malara, 2012; Vinner, 2012; Zaręba, 2012), emphasising the impact of developing the ability to perceive relations between numeric values, which are relations in arithmetic. Generalising the reasoning conducted on particular numbers leads to algebraic thinking, and, in its final form, to using symbolic notation.

Both algebraic thinking and using symbolic notation are issues for students in many stages of learning. This leads to a discussion regarding the ways of teaching algebra. The teacher faces new tasks which require him to engage in different types of activities, as well as creating or modifying educational situations in such a way as to make the acquisition of algebraic knowledge as accessible as possible to the students. In particular, it is necessary for the teachers to have knowledge of the nature of algebraic thinking. It is therefore important to note the results of research of the thought process of a student who generalises and uses symbolic notation. This is the type of research this article refers to. The research itself described in detail in another work (Zaręba, 2012).

Advancing from arithmetic to algebra, which usually takes place at primary school level, usually starts with generalizing by varying constants or by inductive generalizing. The latter begins by observing specific attributes of tasks, and noticing the rules and relations therein. It depends on the teacher whether and how such skills will be developed by the students. As stated by Legutko, teachers *will prefer the development of the skill of generalizing if they can generalise as well.* (...) *They should also have the ability to conduct a discussion with the students regarding them noticing the various rules, ways of writing things down with numbers, letters, algebraic expressions, and equations* (Legutko, 2010, p. 114).

In finding one's way to mathematics and discovering relations and rules, visualisation can be used. The role of this activity has been increasingly prominent in mathematics education literature (Castelnuovo, 1987; Kuřina, 1998; Presmeg, 1999; Tlustý, 2002; Rösken, Rolka, 2006; Yilmaz, Argün, Keskin, 2009; Sochański, 2011; Flores, 2012; Hitt, 2012). It is important for the teacher to notice how the student makes use of it and uses it on the way to generalisation. This knowledge can be used to properly guide the ones who are struggling with mathematics.

2. Terminology - inductive type of generalizing and visualisation

This work was inspired by one of the articles regarding this topic (Yilmaz, Argün, Keskin, 2009). The authors conducted and described studies which allowed me to reflect upon my work (Zaręba, 2012) in the context of noticing the role of visualisation in the process of generalizing. Analysing the results of both works seems interesting due to the fact that the studies described by the authors were conducted independently of one another and used a similar research tool, though aimed at different age groups. The subject of these studies is a specific type of generalizing, i.e. inductive generalizing. This is why the definition of the term as well as a description of visualisation are provided further.

In literature, while defining visualisation, different aspects of it are noted. This article follows the definition made of the authors of the synthetic depiction of visualisation (Hershkowitz, 1989; Zimmermann, Cunningham, 1991; Arcavi, 2003; Rösken, Rolka, 2006; Yilmaz, Argün, Keskin, 2009; Hitt, 2012), including the authors of the article which is referred to further in the text. They note: ***Visualisation*** *is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings* (Yilmaz, Argün, Keskin, 2009, p. 131; in: Arcavi, 2003; Hershkowitz, 1989; Zimmermann, Cunningham, 1991).

The purpose of visualisation is notable e.g. in situations where the method used for a specific case which does not change for a different case can allow a person to understand the rules governing the world of mathematics. Among such situations are e.g. interpreting the law of swapping the places of numbers in a sum (see: Figure 1) or notable product equations presented in a square (see: Figure 2; in: Siwek, 2005, p. 322).

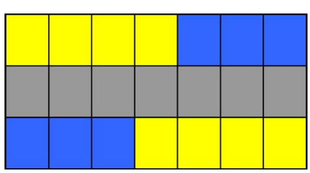


Figure 1. $4 + 3 = 3 + 4$

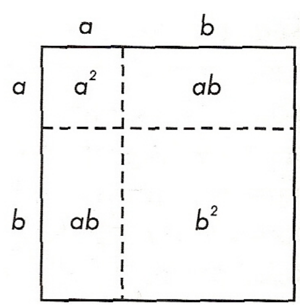


Figure 2. $(a + b)^2 = a^2 + 2ab + b^2$

Generalising of the inductive type is the process of discovering general rules on the basis of observation and comparison of particular cases (Krygowska, 1977; Gucewicz-Sawicka, 1982; Nowak, 1989).

An example of this type of generalizing is presented by Z. Krygowska:

The student is to calculate a sum

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ... + \frac{1}{(n-1) \cdot n}.$$

He begins his reasoning by calculating the first three terms of the equation S_n. The result is:

$$S_1 = \frac{1}{2}, \quad S_2 = \frac{2}{3}, \quad S_3 = \frac{3}{4}.$$

Those are three specific theorems. The student notices that these equations can be a result of the equation $S_n = \frac{n}{n+1}$ by substituting $n = 1$, $n = 2$, and $n = 3$ respectively. The theorem $\bigwedge_{n \in N} S_n = \frac{n}{n+1}$ is a more general theorem than the three specified theorems, as all of them can be a result of the general theorem by specifying it. All of the three specific theorems are certainly true, but the general theorem is, in this stage, still a hypothesis which needs to be studied.

(Krygowska, 1977, pp. 112-113)

A way of merging visualisation with inductive generalisation can be observed in the following task regarding handshakes:

Find the number of handshakes performed when the following number of people meet: 2 people, 3 people, 4 people, 5 people, 6 people, 10 people, and 100 people. How many handshakes will there be, if n people meet?

Here, generalisation manifests in studying the consecutive specific situations of the task (see: Figure 3; Baranowska, 1992, p. 11). All of them regard a specific number of people shaking their hands. The analysis of the procedure or the results (the number of handshakes) leads to solving the task for n people. Depending on the reasoning used, the solution can be presented in multiple ways, e.g. by use of algebraic expressions: $\frac{n(n-1)}{2}$ or $1 + 2 + 3 + ... + (n-1)$.

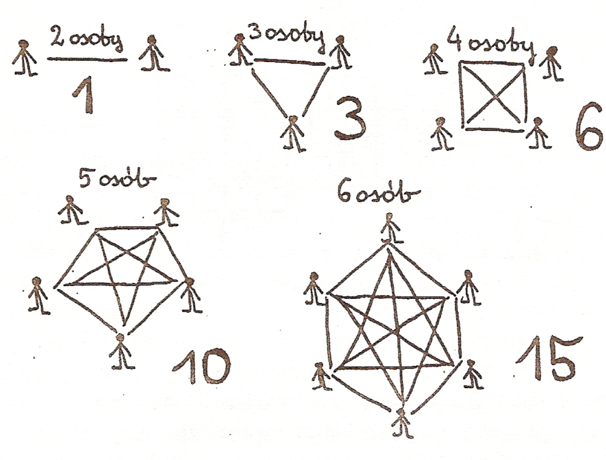


Figure 3. Handshakes

What turns out to be immensely helpful in solving the task is making drawings for the specific cases of the task and their proper interpretation. The role of such drawings in generalizing is outlined further. The description is made in the context of the aforementioned Turkish and Polish studies. The following paragraphs contain insight regarding both studies, as well as the conclusions drawn by the authors and my own thoughts on the matter.

3. Description and conclusions from Turkish research

The Turkish research involved teachers from Turkish upper secondary schools. One of the tasks they solved was this:

How many straight lines can pass through n points
whenever three of them are non collinear?

The problem was not formulated generally for n points at first; it started with two, then three, then four points, gradually increasing until the subject uses generalisation. Thus, the subjects were provoked to perform inductive generalisation. Furthermore, every subject was interviewed, which helped if and how the subjects visualizes and verbalizes while solving the task.

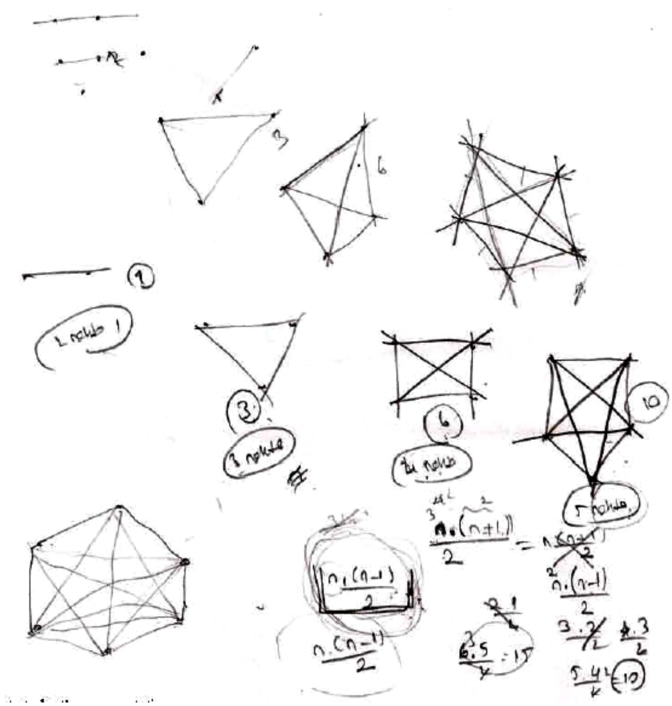


Figure 4. A teacher's solution

The authors describe two representative solutions of the subjects. I outline one of them (see: Figure 4; Yilmaz, Argün, Keskin, 2009, p. 132); provide a short description, the thoughts and conclusions of the authors; I quote selected fragments of the interviews used in the work to justify my interpretation, based on my experience with similar studies of my own[1].

As stated by the authors, the teacher (the subject whose work is presented) used visualisation during her process of generalisation, which helped her discover the proper algebraic expression. The work process of the subject is presented below; crucial steps are supplemented with quotes from the work and own comments.

1. The subject knew that only one line goes through two different points. Despite this, she made a drawing of the situation. In the words of the authors, visual thinking was important for her to make sure her answer was right.

2. In the case of three points, at first she drew them as collinear points, which is when she noticed that they are not. This shows that the drawing helped the subject to find one of the conditions of the task, i.e. the noncollinearity of points.

3. The authors state that in the case of 4 and 5 points, the teacher used visualisation as a way of thinking of possible lines, as the subject visualized the task (which, according to the authors, is indicated by drawing points and lines going through all pairs of points) and found the solution by calculating the drawn lines:

   > When four points were given she directly drew two points and added other two points that are non collinear and then drew the lines by counting. She passed six lines and she was sure about the points. Because she counted and couldn't find anymore lines. So, visualisation was her way to think about the possible lines. And when five points were given the same thing happened.
   >
   > (Yilmaz, Argün, Keskin, 2009, p. 134)

   Taking into account the definition of visualisation provided in the introduction, I think that while the subject did make use of the drawing, I would not define making use of a drawing at this stage as visualisation; it appears as if the drawings are not yet linked to any in-depth thought regarding the solution, rather being used as a tool which helps pinpoint and calculate the proper lines. The drawing stage is, I think, the phase of understanding the task, creating the basis of observation and in-depth thought, which can help focus the subject's attention of different aspects which lead to generalisation. If the subject notices the essence of the general solution in their drawings, I will treat such a drawing as well as the reasoning behind it as a visualisation leading to generalisation.

[1] My interpretation of the work presented by the authors differs to some extent, the interpretation slightly deviating from the one presented by the authors, although I do regard the examples and the analysis presented in the work as very interesting, as well as stimulating reflection and discussion as well as exchanging points of view of research results. I would like to thank the authors of the work for their inspiration.

4. In the case of 5 points, the subject found – as a result of drawing and calculating lines – only 9 (instead of 10) lines. This seems to be a crucial element of her work. This is when the subject develops the need of generalisation, creating a formula. The following is a fragment of the conversation which proves this (S and I are remarks of the subject and observer respectively).

S: /.../ I have nine lines (she thinks). But it mustn't be nine? According to me I must have ten lines.

I: Why?

S: (she restart to count the lines that she drew).one, two, three, ..., nine (she adds the last nine) and ten. I have ten lines.

I: Why did you think that you must have ten lines?

S: I thought to reduce to a formula.

(Yilmaz, Argün, Keskin, 2009, p. 134)

The fact that the subject expected a different result than the one she received from her calculations suggests, in my opinion, that at this point she had already noticed the **arithmetical relation between the received numbers.** She probably had not noticed the relation between the way of drawing the lines and their quantity yet; if that were the case, she would have used a system of drawing lines in order to not omit any. However, in the case of 5 points, the tenth line was missing. The fact that the subject noticed the relation only in the scope of the numbers (instead of both the numbers and the drawing) is also supported by the fact that the authors, when describing the work of the subject, only pay attention to **drawing** and calculating, and not **drawing in a specific way**.

It is worth questioning in what way did the subject create the proper algebraic expression, as well as, particularly, whether visualisation played a role at this stage. Presented below is a fragment of a conversation with the subject related to the discovery of the formula as well as two interpretations (the authors' and mine) in the context of the use of visualisation at this stage of her work.

I: O.K. Can you reduce a formula?

S: Hmm ... Now ... **I passed one line from two points**. (she draws again two points and a line and write 1 near the picture). Then I passed three lines from three points. (she draws again and write 3 near it). If I think four points (she draws again four points and **draws the lines by counting**) one, two, .. and six. O.K. I have six lines (and she writes 6 near the picture). So, If I think five points (**she draws the same picture by counting**) one ... ten. I have ten lines. (she writes 10 near it and then she controls the pictures and thinks) **one from two points, three from three points, six from four points, ten from five points.** Hmm ... I must formulate it with n. (she thinks) I think to have a sequence and then to reduce to a formula. (she writes the numbers of lines and thinks). **I pass a line from two points** (writes $\frac{n(n+1)}{2}$ and **tries the numbers on this formula**). **Three lines from three points**, so, I must write something in the parenthesis that must simplify the number 2, O.K. It must have 2 in it (she thinks and writes $\frac{n(n+1)}{2}$ and thinks the numbers) no, this isn't (she crosses out that she wrote and thinks). If I write $\frac{n(n-1)}{2}$?

I: Why did you do this?

S: When I try these points; one from two, three from three .. (she tries the numbers of the points on the formula). The formula works for five points. If I try for six points (she writes for six points and has 15). I have 15 lines. ??? Is it true? Hmm... (she draws six points and passes lines from them by counting) one, two, ... fourteen, fifteen. Yes, I have 15 lines. It is true.

(Yilmaz, Argün, Keskin, 2009, p. 134; own emphasis)

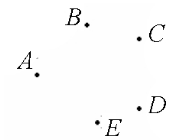
The authors, noticing the useful role of visualisation in the generalisation, interpret the conversation as follows.

> When reducing the formula, the denominator of the formula was 2. She recognized that a line passes through two points. So she put 2 in the formula. Then she thought $n(n+1)$ for the part of remain and she tried the numbers on this. The lines that she drew helped her to recognize that the formula was not true. Then she thought $n(n-1)$ and she tried again and found the formula. Also **during this time visualisation helped her to find the formula**.
>
> (Yilmaz, Argün, Keskin, 2009, p. 134; own emphasis)

The authors emphasize that the subject made drawings and calculated lines, but my concern is whether the subject noticed anything more in the way she drew the lines, whether there was any thought process put into this, whether dividing by 2 and creating factors n, $n-1$, or $n+1$ was just a way of matching the equations to the received numbers in such a way as to make the whole formula work for the specific numbers which were the solutions in their specific cases, or whether the role of visualisation here did not only consist of making a drawing, but also on "seeing" the elements of the formula in the drawing. It is my understanding (considering the definition of visualisation outlined earlier) that **visualisation is the art of looking, but also seeing.** In this task, this would indicate seeing the general formula based on the visual representation. The visualisation could then be considered a vessel for the general solution. However, in the described work it seems that despite the drawings, the subject did not make much use of them: her comments concerned strictly the numbers received, which she compared just before attempting to create a formula: "**one for two points, three for three points, six for four points, ten for five points.** Hmm... I need to formulate this using n." Therefore, she was analysing the numbers and trying to cover them with a formula with the n variable. It is sensible to think of visualisation in the context of one of the remarks of the subject that **the denominator has to be 2, as one line goes through 2 points.** However when taking the entire remark into account, it does not have to indicate a general observation which was used in further reasoning. The subject did not state that a line goes through **all** pairs of points; she could have only noticed the fact that the first solved case (of two points) the answer was one. This number is twice as small as the variable n, which in this case equals 2. This could suggest dividing by 2 in the sought-after formula.

The role of visualisation in the presented task could be presented in the reasoning shown for the 5-point case in Figure 5.

In conclusion, it is my point of view that in the example analysed above, visualisation does not play a significant role in the process of generalisation. While a drawing was indeed used, as the task is of a geometric nature, the subject created the drawing in order to find the numerical results in specific cases of the task. Creating the general formula did not, however, involve the drawing. The subject found the formula by adapting algebraic expressions to the numbers received. The reasoning of the subject lacks an important element of visualisation – reflecting on the drawing. This thought process – in terms of the definition of visualisation used by me and the authors of the article – should lead to *discovering new ideas, new knowledge, and deeper reasoning.*



I'm drawing lines defined by:
point A and every other point: line AB, line AC, line AD, line AE
point B and every other point: line BA, line BC, line BD, line BE
point C and every other point: line CA, line CB, line CD, line CE
point D and every other point: line DA, line DB, line DC, line DE
point E and every other point: line EA, line EB, line EC, line ED.

Every point (of which there is 5) connects to every other point (choosing 1 point leaves 4 other points).
Every line repeats itself twice, since line AB is the same as line BA, line CD is the same as line DC, etc.

Therefore I have $\frac{5 \cdot 4}{2}$ lines.

Figure 5. Solution involving the use of visualisation

In my opinion, the role of visualisation is reflected very well in the second example presented by the authors. In the cited work, the subject gets the following numbers in consecutive cases (2, 3, 4, 5, and 6 points): 1, 3, 6, 10, 15; she notices that the number of lines increases in the following case by 2, 3, 4, and 5 when compared to the previous case. Searching for a general equation, she finds the visual justification of the growth:

S: /.../ (he adds one point to the picture for four points by drawing). When I add one point, one point is added for each one. When I add one point to 4 points, 4 lines are added. I mean that 4 lines are added to the number of lines for 5 points. So, I think, $n-1$ lines will be added to the number of line for $n$points. I must know $n-1^{th}$ to know n^{th}./.../

(Yilmaz, Argün, Keskin, 2009, p. 135)

Situations like this combine the algebraic equation with its graphic representation. The subject creates a drawing which ends up being thought-provoking. She notices more components of the desired equation while drawing.

The two approaches of the subjects presented below clearly show the difference between visualisation and a drawing which is not used for generalisation. This difference is outlined in the next paragraph. While describing the solutions of students contained therein, I detail the solutions which are related to a properly made drawing or chart. However, I emphasize the role of the solutions which contain both a drawing and a thought process related to the drawing.

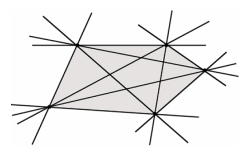
4. Description and conclusions from Polish studies

The Polish studies described herein were aimed at 13-14 year old middle school students. Their main goal was the description of the process the students go through to perform inductive generalisation and write it down by using symbolic notation. By analysing these studies, the occurrence of visualisation can be pinpointed alongside its role in creating a symbolic generalisation. The students were solving various tasks during individual conversations; I will focus on a problem similar to the one related to the Turkish studies. It is clearly visible in Figure 6. The solution, in the case of n lines, leads to the same formula as in the case of the Turkish studies, i.e. $\frac{n\cdot(n-1)}{2}$, or alternately: $1+2+3+...+(n-1)$.

1. A convex pentagon is given. Every two vertices of this polygon define one straight line.

   Were all the possible straight lines drawn? ....

   What is the number of possible straight lines? ....



2. Draw a triangle and a convex quadrilateral.
   In both cases draw all the straight lines defined by each pair of vertices of the figure.

   How many such lines are defined by the vertices of a triangle?... Of a quadrilateral?...

3. Fill in the table.

Number of the angles of a polygon	Number of straight lines defined by each pair of the polygon's vertices
3	
4	
5	
6	
7	
10	
24	
345	

4. Let n denote a number of vertices of a polygon. How can the number of the straight lines defined by this n-gon's vertices be described?

Figure 6. Research issue of Polish research

By solving the tasks of the card, the students advanced through subsequent specific cases: for $n = 2, 3, 4$, etc. In every one of them, they created a certain situation model. Solving the problem within the framework of a particular model is connected with an interpretation based on a produced sketch or arithmetical scheme. In the work-sheets of the participants I have distinguished six mathematical models. Due to the size of the article, only three will be described in more detail – those which are relevant to the concept of visualisation. The description will be based on fragments of the work of one of the students. Other models will only be acknowledged.

1. **Geometrical model (G)** – a pupil indicates lines at the drawing and arrives at the solution by counting the previously indicated lines (see: figure 7).

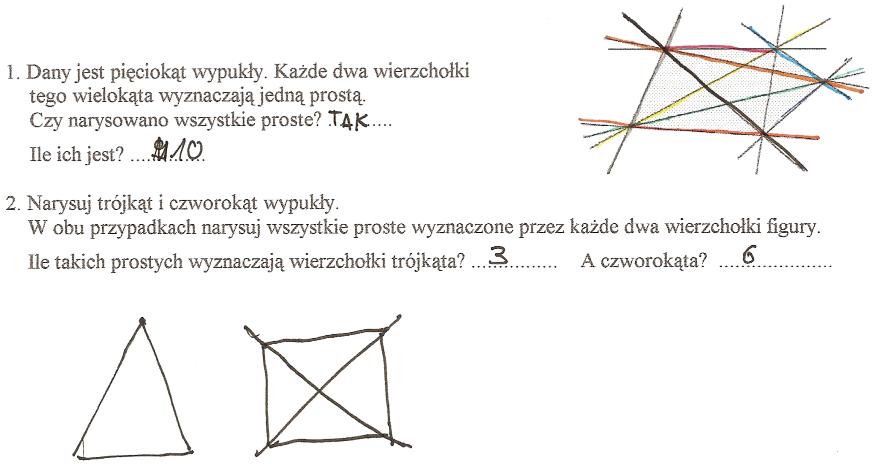
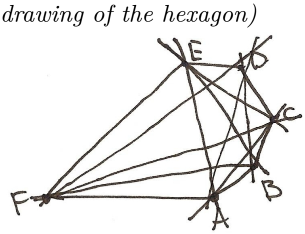


Figure 7. Excerpt of the work of the Polish student

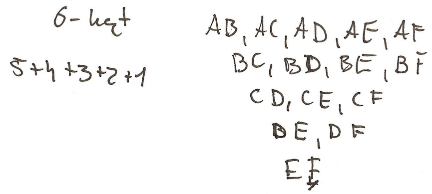
2. **Geometrical–arithmetical model (G-A)** – in order to find the number of lines in a given particular case a pupil indicates them in a way which guaranties that no lines has been omitted. He uses this method to create the arithmetic expression which serves as the solution in the case of a particular polygon.

O: For the hexagon, you didn't know whether you had drawn all the straight lines. **Let's try to find a method that would allow us to state that all the straight lines are already drawn**.

S: I think I know. Let's say, for the hexagon.
(He points at the drawing of the hexagon)



S: *(The student creates and makes comments on the undermentioned note). AB, aha*, with A. That would be: AB, AC, AD, AE, AF. And now for B: BC, BD, BE, BF and that's all of them. And now for C: CD, CE, CF *(he finishes in silence).*



S: I've noticed that at the beginning I got 5 straight lines and this number decreased. It was 5, 4, 3, 2, 1.

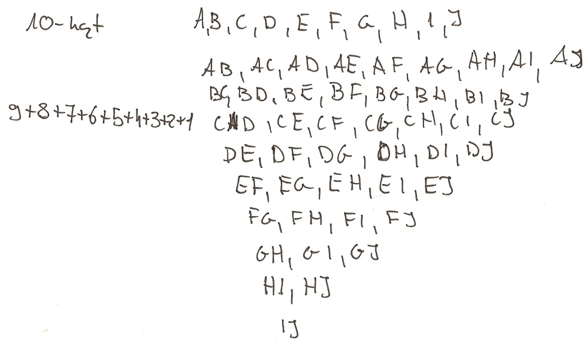
O: So, **how can this number be obtained?**

S: $5 + 4 + 3$ makes 12...

O: **I don't want to know the number, but the method.**

S: $5 + 4 + 3 + 2 + 1$.

For a decagon



S: On the beginning, I had 9, so $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$.

Visualisation in the cited work is not just the drawing, which is used to calculate the lines; there is a general blueprint of a solution to be observed for every case of the task – this blueprint or drawing allows for the generalisation to be observed in the form of a formula. A specific layout of the lines (in the form of an "upside-down pyramid") allows to observe a sum, the subsequent components of which present the number of lines in a particular location of the layout.

3. **Arithmetical model (A)** – the student identifies the number of lines with a respective arithmetic expression without appealing to geometric interpretation of the solution of this problem.

S: *(When the number of the angles of a polygon is 24)* I think I'm not going to write all the numbers down, it might be done in a shorter way, but how...

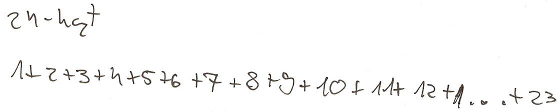
O: If you do it this in a way you have here *(for the decagon)* – would it be OK?

S: Yes, but in the case of a polygon that has 345 interior angles, I would need lot of sheets.

O: Maybe you could give up something; take a look: for the hexagon you have been still drawing. For the decagon you've given up drawing – and you have less work to do. And now – what could you give up to make it even shorter?

S: Aaaah! *(flash)* **I only need to use this expression (he points at the sum of the numbers from 9 to 1). I mean, if there were nine of**

these numbers, I can write: 1 +2 + 3, and so to 9. /.../ It will be 1 + 2 and dots and + 23.



O: And **for 345-gon?**

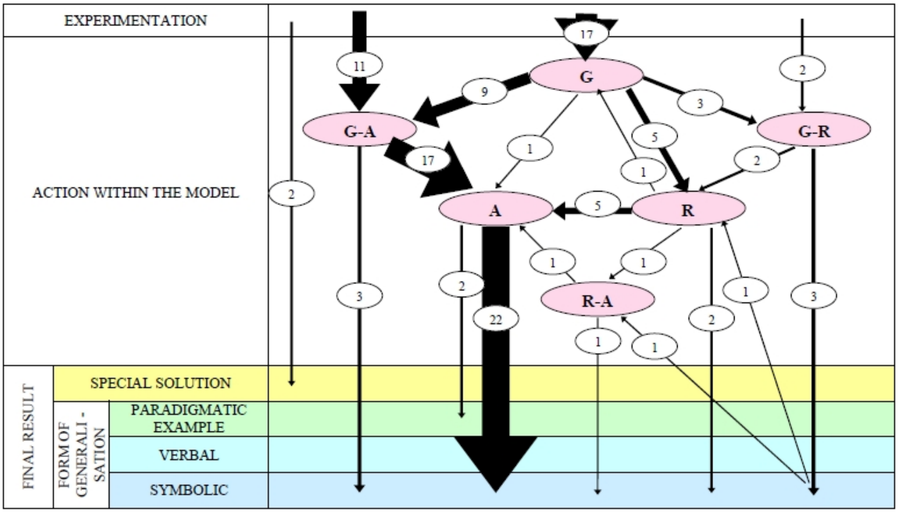
S: It will be ***(he writes the sum)***.

To observe the application of the models described above in the full works of the subjects, I will point out the three remaining models. They are related to the perception of a recursive relationship.

4. **Geometrical-recursive model (G-R)** – every time after drawing the next point, the student "sees" new – when compared to the previous situation – lines. This corresponds to the second solution presented in the article described above.
5. **Recursive model (R)** – while analysing (in isolation from the sketch) the number of lines for a few particular cases, a pupil notices a recursive relationship between them, notices the consecutive increase by 2, 3, 4, ... of the number of lines with respect to the number of lines for the previous polygon.
6. **Recursive-arithmetical model (R-A)** – the student notices the recursive relationship (that is, he uses the **R** model), notices how the number of lines increases subsequently by 2, 3, 4,... in comparison to the "previous polygon," and he creates a formula based on this relationship, which allows to find the number of lines for a given polygon without having to refer to its previous case.

The analysis of the behaviour of the subjects suggests that activity related to mathematical models plays an important role in the generalisation process. In particular, before performing generalisation and writing it down using symbolic notation, the student's reasoning makes use of a specific sequence of models used. I have noted the aforementioned models in the work of every student and indicated, by using arrows on the drawing, the process the student goes through from the moment of familiarizing himself with the task (the experimentation phase), through working with various models, to solving the task and performing generalisation. Scheme 1 contains various types of solutions (task solutions only for specific cases, solution in the form of a paradigmatic example, and generalisation in verbal or symbolic form; see: Zaręba, 2012, p. 130; own translation). In the context of this work, the arrows relevant to the task are those pointing at the solution in a symbolic form. The numbers on the arrows represent the number of people who advanced from the model where the arrow begins to the one the arrow is pointing at.

Scheme 1 shows that the activity of all subjects begins in model **G** or a related model (**G-A** or **G-R**). It is worth noting that model **G** is, in my opinion, unrelated to visualisation, as the link between the drawing and the suitable numeric expression is established by models **G-A** and **G-R**; the most common path (**G**→ **G-A**→ **A**) indicates that model **G** is simply the introduction to **G-A** model reasoning, in which the activities of most subjects gradually deviate from geometry and move mostly towards model **A**, which is a scheme with use of numbers. This is one step away from generalisation and creating a formula containing a variable. The students transition from a given arithmetic expression to an algebraic one.



Scheme 1. Synthesis of Paths Leading to the Generalisation

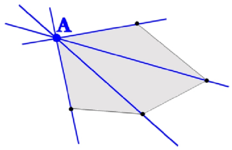
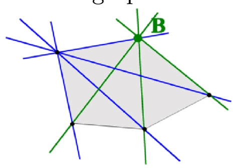
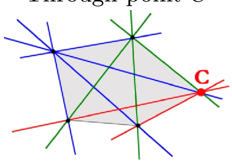
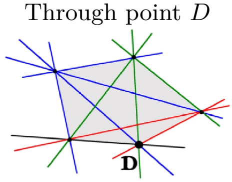
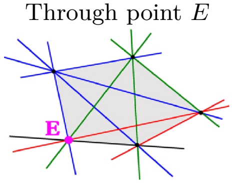
The most effective way is by using model **G-A** which links the student's drawing to the arithmetic expression. This is the stage of the student's work in which I can see a lot of potential, as this is the stage which involves the use of visualisation in generalisation. It seems that this is the path the students should be encouraged to take – so that they can "see" the proper relationships and the formula in the drawing they're looking at.

5. Conclusion

The authors of the works described in the article refer to the role of visualisation in the process of generalisation. The definition of visualisation is the same among them, but their interpretations differ in regards to solutions which use schemes or drawings. This is reflected in the analysis of the first solution presented in the article describing the Turkish research. The Turkish authors are of the opinion that the subject uses visualisation, while the author of the Polish work does not acknowledge this. It seems as though the student's drawing is not coupled with an appropriate thought process. The reflecting should be based on the drawing, not on the numerical result on which the drawing was based. In the first case the formula of the identified relationship can be seen in the drawing, whereas in the second case the drawing is not directly related to the identified relationship. Furthermore, different outlooks on the same process show how difficult it is to understand the thought process of someone who is solving a given task, and how important it is to properly interpret the verbal and non-verbal messages of the person.

In the case of the second solution described by the Turkish authors, there is no discrepancy in the interpretation of visualisation. This is why in the scope of the work where drawings were used to "see" the appropriate formula, both the Turkish (regarding adults) and Polish (regarding students) research shows that visualisation is helpful in the process leading to generalisation. The analysis of the paths taken by the students leading to generalisation seems to suggest **how to organize the didactic process aimed at developing this activity, especially if the generalisation takes on a symbolic form with use of a variable.** Taking the presented research into account, it is desirable to **suggest the way described above: from a geometric view of a problem to an arithmetical one**, although a smooth

Marta counted the number of lines defined by the vertices of a pentagon. Read her notes thoroughly. Explain her method.

	How many lines
Through point A	
	4
Through point B	
	4 + 3
Through point C	
	4 + 3 + 2
Through point D	
	4 + 3 + 2 + 1
Through point E	
	4 + 3 + 2 + 1 + 0

There are 4+3 + 2 + 1 + 0 lines.

Figure 8. Visual hint

transition between them is key. Therefore, it would also be desirable for the first tasks of a tool designed to develop this type of activity to have visual representations, and for models of a geometric character to be suggested by the teacher during a student's initial steps. Suggestions which direct the student towards using a drawing to "see" the relationships also seem important. For instance, in the Polish research described above (regarding the number of lines defined by all pairs of vertices of a given polygon), the following suggestions and questions proved to be valuable:

- Let's find a method that would allow us to state that all the straight lines are already drawn.
- So, how can this number be obtained? (when the student obtains a solution by reading it from the drawing)

- I don't want to know the exact number now, but the way to obtain it.
- Geometrical directions facilitating perception of arithmetic relationships. Exemplary suggestions are presented in Figures 5 and 8 (see: Zaręba, 2012, p. 88; own translation).

The suggestions presented above are just some of the ways the students can be directed towards generalisation. I think the potential is in the students themselves, and the role of a good teacher is to listen to the student and to arrange their thinking in such a way so that they can make use of their potential.

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