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# Intuitive explanations of mathematical ideas*

**Abstract.** This short note is devoted to the role played by intuitive explanations in mathematical education. We provide a few examples of such explanations. They are related to: verbal commentaries, perception, physical models. We recall also some examples of internal explanations, inside mathematics itself.

## 1. Introductory remarks

It should be stressed at the very beginning that all reflections that follow are based on the author's experience of teaching mathematical logic and introduction to mathematics at the university level only. The author does not have any experience in teaching mathematics at the school level. We nevertheless hope that the remarks below may be of some significance to mathematical education in general. It is of course natural that much effort in mathematical education studies is devoted to teaching of young pupils. We think that more attention should be paid also to the mathematical education of adults. A few years of their presence at the university offer in most cases the last chance for developing and mastering these mathematical skills which are so profitable in various situations, both professional and private. Especially in the case of the students of humanities these years offer the last possibility of overcoming the math anxiety so typical for such students. Thus, we claim that teaching mathematics to the students of humanities may play a therapeutic role. This teaching should avoid boring drill exercises and focus on the presentation of really interesting mathematical problems. If we are successful in these attempts, then the students are no longer afraid of mathematics, they

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start to appreciate mathematical reasoning and become aware of the role played by mathematics in culture.

There is an important difference between pupils and adults concerning the role of intuitive explanations. Namely, it is possible to make the latter aware of the distinction between formal and intuitive exposition of mathematical ideas, which is not so clear to the young pupils.

Good example of what we mean by *intuitive explanation* is provided in the following quotation from Poincaré's *Science and Method*:

> We are in a class of the fourth grade. The teacher is dictating: 'A circle is the position of the points in a plane which are at the same distance from an interior point called the centre.' The good pupil writes this phrase in his copy-book and the bad pupil draws faces, but neither of them understands. Then the teacher takes the chalk and draws a circle on the board. 'Ah', think the pupils, 'why didn't he say at once, a cir -cle is a round, and we should have understood.'

We quote this fragment from (Sierpińska, 1994) (*Understanding in Mathematics*, p. 1). Each chapter of the book starts with a quotation from Poincaré. Notice that the shape of the circle depends on the distance function: circles in Euclidean metric look differently from, say, circles in Manhattan metric or Tchebyshev metric.

We use the very term *explanation* in a popular sense and not in the sense attributed to it in the general methodology of sciences. The latter, theoretical approach to *explanation in mathematics* is discussed for example in: (Steiner, 1978), (Kitcher, 1984), (Hanna, Jahnke, Pulte, 2010), (Lange, 2014), (Mancosu, 2015). Here we focus our attention on the heuristic role of intuitive explanations.

The meanings of mathematical concepts are established in the underlying theory. Recently accepted standards require that a mathematical theory should be formulated on the axiomatic basis. However, in order to fully *understand* particular mathematical concepts we must mentally associate these meanings with a series of intuitive representations.

We accept proposals of Anna Sierpińska concerning the notion of *understanding* in mathematics – cf. Sierpińska 1994. The author accepts Ajdukiewicz's theory of meaning and understanding, cf. his formulation: *a person understands an expression if on hearing it he directs his thoughts to an object other than the word in question* (Ajdukiewicz, 1974, p. 7). Some of her most important ideas may be summarized as follows:

1. Sierpińska extends Ajdukiewicz's proposal so that not only expressions may be understood, but also problems. Thus understanding is a relation holding between *object of understanding* and *basis of understanding*
2. The mathematical objects are *seen as* something and not plainly *seen*.
3. The *objects of understanding in mathematics* include: concepts, problems, formalism, text.
4. The *basis of understanding* includes: representations (e.g. mental images or conceptual representations, procedural representations), mental models, apperception, thoughts that (so and so).

5. *Mental operations* involved in understanding include: identification, discrimination, generalization, synthesis.
6. *Attention* and *intention* are necessary conditions for the act of understanding.
7. The author points to the role of *epistemological obstacles* in the process of understanding.

About the role of explanatory procedures in mathematics Sierpińska writes (Sierpińska, 1994, 76):

> The quest for an explanation in mathematics cannot be a quest for proof, but it may be an attempt to find a rationale of a choice of axioms, definitions, methods of construction of a theory. A rationale does not reduce to logical premisses. An explanation in mathematics can reach for historical, philosophical, pragmatic arguments. In explaining something in mathematics, we speak *about* mathematics: our discourse becomes more metamathematical than mathematical.

We suggest that the main concept of the present paper, i.e. *intuitive explanation* may be given the following characteristic:

1. *Intuitive explanation* is a relational concept. It is understood here in a pragmatic sense (clarification of ideas, heuristic methods, hints facilitating understanding).
2. Purpose of intuitive explanations is clear: they are responsible for evocation of understanding.
3. Tools used in intuitive explanations: paraphrase, translation, analogy, metaphor, model building, diagrams, etc. We are going to provide some examples below.

### 1.1. Three contexts

In the general methodology of sciences there is a clear distinction between two contexts: that of *discovery* and that of *justification*. In the case of mathematics they are discussed widely in the literature: cf. for instance (Lakatos, 1976), (Polya, 2009). We propose to add a new context, i.e. that of *transmission* (Pogonowski, 2016). All three contexts can be shortly characterized as follows:

1. *Context of discovery*: all ways to mathematical discovery, most notably mathematical intuition. Mathematical discovery can not be described algorithmically. Imagination, insight, reasoning by analogy, inductive hints, heuristic thinking, aesthetic values and other factors may be involved in the process of discovery.
2. *Context of justification*: proofs based on deduction. It is sometimes claimed that a mathematical proof is a confirmation of intuition which was responsible for discovery of the result in question. Mathematical proof is based on logical consequence but in practice it differs from a purely formal proof as understood in formal logic. Some steps in a mathematical proof may be omitted (as self evident or trivial).

3. *Context of transmission*: talking *about* mathematics, teaching it, and its popularization. Creating of mathematics and practising it is a matter of the two contexts mentioned above. Learning mathematics, teaching it and popularizing it to the wider public – these activities embrace relationships between mathematical objects, theorems, procedures on one side and human subjects on the other side.

Intuitive explanations are the most important components of the context of transmission. We think of such explanations as tools appropriate for domestication of mathematical ideas. They connect abstract meanings from a mathematical theory with suitable mental representations. Students may have very diversified intuitions at the beginning of their mathematical study. In the processes of learning and teaching mathematics these preliminary intuitions are supposed to become unified and should correspond to the prescribed standards.

Let us consider two examples showing that even famous mathematicians and physicists require essential support from intuition for the understanding of mathematical ideas:

1. Georg Cantor proved that the sets $\mathbb{R}$ (points on the line) and $\mathbb{R}^2$ (points on the plane) are equinumerous, informed about this result his friend Richard Dedekind and wrote: *Je le vois, mais je ne le crois pas!* (I see it, but I do not believe it). Here *see* obviously refers to the fact that the result has obtained the rigorous *proof* and *believe* refers to the reader's (and many of his contemporaries as well) *intuitions*. The result forced anybody to make change in beliefs accepted on an intuitive basis: it has been shown that cardinality is independent of dimension.
2. Anna Sierpińska (Sierpińska, 1994, 36) recalls a confession made by Heisenberg in his *Der Teil und das Ganze* (1969), where he complained about his troubles with understanding of the relativity theory: *Ich habe die Theorie mit dem Kopf, aber noch nicht mit dem Herzen verstanden* (I have understood the theory with my head but not yet with my heart). This case resembles the previously discussed situation of Cantor and Dedekind.

## 1.2. Mathematical intuition

A few words are in order concerning the very term *mathematical intuition*. We propose to distinguish several sorts (or levels) of mathematical intuition:

1. *Protointuitions*. These intuitions are related to our cognitive abilities. We think that examples of such intuitions are: the ability to approximate the size of small collections, the ability of subitizing, drawing distinction between inside, outside and boundary, etc. Protointuitions are the object of study of cognitive psychology and cognitive science.
2. *Intuitions developed in the school*. These are numerous and include, among others: beliefs concerning arithmetic operations, mental images corresponding to geometrical constructions, intuitive ideas about order, metric, probability, etc.

3. *Intuitions of professional mathematicians.* These beliefs strongly depend on individual talent as well as on the research experience of mathematicians. It is a challenge to reconstruct such intuitions from the source texts. Reports from introspection may also be helpful, but they are not numerous.

We are not going to discuss here several views on mathematical intuition dispersed in the philosophical works. The views of Descartes, Kant, Poincaré, Hadamard, Gödel and other prominent mathematicians and philosophers have been extensively discussed in the literature. We appreciate opinions on mathematical intuitions formulated by Davis and Hersh (Davis, Hersh, 1981). In particular, they formulate their own position with respect to mathematical intuition referring to the processes of learning and teaching mathematics. Thus, mathematical intuition is not a direct perception of something existing externally and eternally. Rather, it is an effect emerging in the mind after numerous experiences with concrete objects (including signs and mental representations).

Let us also point to the following opinion concerning the role of intuition in mathematics (Fischbein, 1987, p. 201):

> In our opinion intuition is the analog of perception at the symbolic level. It has the same behavioral task as perception, namely to prepare and to guide our mental or practical activity. Therefore an intuitive conception must possess a number of features analogous to that of perception: globality, structurality, imperativeness, direct evidence, a high level of intrinsic credibility. In this way intuitions are able to inspire and guide our intellectual endeavors firmly and promptly even in a situation of uncertain or incomplete information.

From the point of view of mathematical education it is of uppermost importance to have some precise criteria which can be used for distinguishing between good and bad intuitions. Among the didactic goals in mathematical education one usually mentions the following:

1. Developing calculation skills.
2. Developing abilities for problem solving.
3. Developing abilities of mathematical modelling.
4. Developing (or shaping) mathematical intuitions.

The most difficult seems to be the last task. In the case of the first three tasks we may propose several clearly described procedures and strategies, often even in algorithmic form.

As it happens, people may have very bizarre images or representations concerning the behavior of physical bodies. For example, as Talia Ben-Zeev and Jon Star write (Ben-Zeev, Star, 2001, p. 29):

> It has been found that when people were asked to draw the path of a moving object shot through a curved tube, they believed that the object would move along a curved (instead of a straight) path even in the absence of external forces.

Such views are classified as belonging to *folk physics* (also called *naive physics*). The views in question are very often simplifications or misunderstandings of real phenomena. They are investigated in cognitive psychology.

## 2. Examples

Our examples are not collected in a systematic manner. We have simply selected a few such examples, referring to verbal explanations, appeal to perception, construction of physical models and cross-domain explanations inside mathematics itself.

### 2.1. Verbal explanations

Mathematical texts are written in a language being a mixture of a symbolic language and a natural one. By a verbal explanation one usually means a reformulation of symbolic formulas in a colloquial language. However, such a reformulation is not a mere translation but it contains also some intuitive comments.

1. *Sets, relations, functions.* When talking about sets we can – in simple cases – describe them as *containers*, *boxes*, *catalogues*, etc. consisting of some elements. However, attention should be paid to at least two things. First, set theory does not make any use of sets consisting of some physical individuals. Sets in general are abstract objects satisfying the axioms of set theory. Talking about sets of some real (physical) objects is ubiquitous and it is justified to the same degree as any other application of mathematics in the description of real phenomena. Second, set theory originated in the investigation of rather complicated collections of real numbers to which the simple container-metaphor is hardly applicable.
   It is also common to associate a dynamic interpretation to the concept of function in the sense that one proposes to understand functions as processes. One has to remember that such an interpretation is strongly based on an intuitive understanding of the dependence of values of a function on its arguments. From the purely formal point of view the fact that a function *maps* objects from its domain into objects from its range (or transforms the former into the latter) does not directly support any process-like understanding of this concept. Functions (and relations as well) should be ultimately described in an extensional manner, as sets.
2. *Calculus.* The ordered field $\mathbb{R}$ (with norm based on absolute value) may be considered as sufficient for developing Calculus. We make use of its algebraic and arithmetic properties (a number field), its order properties (continuous natural ordering compatible with field operations) and its topological properties (natural topology on $\mathbb{R}$). What linguistic tools are used when talking about the fundamental concepts of Calculus? We limit ourselves to a few comments:
   (a) *Real variable.* It is customary to talk that the independent variable somehow *moves* in the domain. Thus the student may think that it

walks, runs, flies, swims (but not jumps) in this domain. All these metaphors are innocent as long as we remember that under the linguistic usage there is a precise meaning: the variable accepts (or takes) particular values in the domain.

(b) *Limits.* It is customary to say that members of a sequence *tend* to its limit (if it exists). They *approach* the limit, are *arbitrarily close* to it. This is precisely expressed in the $\varepsilon$-$\delta$ setting, where closeness is characterized in terms of the absolute value.

(c) *Continuity.* The fact that a function is continuous at a given point is linguistically expressed by saying that whenever a sequence of arguments tends to this point, then the sequence of values of these arguments tends to the value of the function at that point (Heine definition). Again, this is precisely expressed in the $\varepsilon$-$\delta$ setting, where closeness is characterized in terms of the absolute value (Cauchy definition).

(d) *Derivative.* This concept is discussed with a reference to the *rate of change.* It is also clarified by the reference to geometry (tangent to a curve at a given point).

(e) *Integral.* This concept is related to *summation.* It is customary to clarify it with a reference to *area* below the graph of a function.

Let us add that verbal explanations concerning concepts of Calculus may contain certain redundancy which has an essential value for better understanding of such concepts. Suppose that we want to express verbally the formula:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x\ \forall y\ (d(x, y) < \delta \rightarrow d(f(x), f(y)) < \varepsilon)$$

Most likely, our reading would be as follows:

> For an arbitrarily *small* positive number $\varepsilon$ there exists a positive number $\delta$ such that: if the distance $d$ between the arguments of the function $f$ is less than $\delta$, then the distance between the values of $f$ for these arguments is less than $\varepsilon$. In other words: if the arguments of the function are sufficiently near, then the values of the function for these arguments are arbitrarily near. In addition, "sufficiently near" is understood here in the same way for any choice of arguments (uniform continuity).

Observe that in the above formulation the qualification *small* is added to the formula itself but it may be considered as helpful for better understanding of the formula in question.

3. *Metaphors.* Verbal explanations can make use of metaphors but in this case one should be very careful. We do not adhere to the approach to mathematics proposed by Lakoff and Núñez in their already famous monograph (Lakoff, Núñez, 2000). In particular, we doubt in the explanative power of the Basic Metaphor of Infinity formulated by these authors. According to it we should be able to complete each iterative step-by-step process with its ultimate end. The authors claim that in such a way one obtains all infinitary objects in

mathematics: infinite sets, limits, closures, etc. We disagree that one should understand a circle as a limit object of a sequence of regular polygons – a circle is an object which is defined in terms of distance, as it is well known. It is not difficult to define the relation between two series holding when one of them is, say, more slowly divergent than the other. According to the Basic Metaphor of Infinity one should be then able to form a limit object: the most slowly divergent series. But such a series does not exist, as it is well known. To sum up: metaphors may play auxiliary role in explanations but they should be applied in a reasonable manner, consistent with mathematical laws.

In can be added that the mathematical terminology itself contains intuitive hints as far as the meaning of the concepts in question is concerned – cf. e.g. such terms as: *dense* sets or *nowhere dense* sets, *isolated* points, etc. The names of theorems and lemmas can also contain some intuitive information – cf. e.g. the *overspill lemma* in arithmetic.

### 2.2. Perception

Most common and important intuitive explanations in mathematics related to perception are connected with vision. Touch and audition play a secondary role.

1. *Diagrams.* Pictures, drawings, diagrams, schemas, etc. are not accepted as legitimate methods of proof but they may clearly have a great explanatory power. However, one should follow several rules in this respect: diagrams should not be too much suggestive, the conventions according to which one gets a graphical representation should be consistent and explicitly given (say, in representing a three-dimensional object on a piece of paper). The book (Needham, 1997) is an excellent example how adequately chosen pictures and diagrams can help in understanding even an advanced mathematical theory (in this case: complex analysis). Illustrations play very important role in an introduction to algebraic topology, cf. e.g. (Prasolov, 2011). An interplay between diagrams and algebraic formulations is clearly visible e.g. in (Adams, 2004) (introduction to knot theory).
   Colors are commonly used in elementary textbooks. However, they appear also useful in the books on more advanced mathematics, notably in graphical representations of analytic functions (different colors are associated with different values of the modulus of a function). Color illustrations are expensive in the printed text. In texts produced electronically, as well as in computer animations colors are "for free" and hence may be extensively used.
2. *Geometrical representations of numbers.* Introduction of numbers in early education is based on experiments with physical objects. At later stages one can explain certain properties of arithmetical operations by reference to geometry, considering e.g. *triangular numbers*. Many interesting examples of this sort can be found e.g. in (Conway, Guy, 1996).
3. *Rubber-like objects.* It is very common in textbooks devoted to topology to compare general topological spaces with rubber-like objects. This comparison

is justified by the properties of homeomorphisms, i.e. continuous bijective mappings with continuous inverse between topological spaces which preserve closeness.

4. *3D-models*. Wooden, steel-made, plastic or plaster objects, representing solids discussed in the school, like cone, cylinder, ball, cube, etc. are commonly known and very useful in teaching. In (Lénárt, 1996) one can find a description of a set of tools which may be used, among others, to explain concepts from non-Euclidean geometry.
5. *Movies and software*. A lot of new possibilities in mathematical education is provided by the use of movies and software, which in many cases are freely accessible on the web. There are numerous lectures or lecture series devoted to particular domains or theories, recordings of panel discussions, etc. But more importantly, one can speak of a new tendency in teaching mathematics – the one based on (usually short) movie clips discussing a chosen topic. One can justly speculate that this form of teaching will be constantly growing. This is partly because for the contemporary students the first place where they look for information is the internet.
   Mathematical software such as *Matlab* or Wolfram's *Mathematica* offer recently really incredible possibilities not only for the research work itself but also for learning and teaching mathematics. Let us also notice that some textbooks even demand to solve exercises described as: make an experiment with a computer (meaning: check something using the software). The free software *GeoGebra* also offers a lot of interesting tools to be used in teaching arithmetic, algebra and geometry.

## 2.3. Physical models

The use of physical models in mathematical argumentations has a very long history. Archimedes, in his famous *The Method*, describes a method to determine volumes which involves balances, centers of mass and infinitesimal slices. He is fully aware that such argumentation based on mechanic is not a genuine mathematical proof. He then provides strict proofs, using the method of exhaustion. To quote Archimedes himself (the text of *The Method*, (Heath, 2002, p. 13–14)):

> This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.

1. *Linkages*. (Ghrist, 2014) contains numerous interesting examples showing the interpretation of topological notions. Let us limit ourselves to one example only. Ghrist recalls a theorem saying that *Any smooth compact manifold is diffeomorphic to the configuration space of some planar linkage* and adds a comment (Ghrist, 2014, p. 12):

> This remarkable result provides great consolation to students whose ability to conceptualize geometric dimension greater than three is limited: one can sense all the complexities of manifolds *by hand* via kinematics.

Linkages themselves have a great and interesting history. Archimedes, Hero of Alexandria, Leonardo da Vinci, were precursors of the investigations of the connection between geometry and mechanics. Linkages still play essential role in engineering: in the construction of engines, robots, etc. From a mathematical point of view, they are related to the problems of transferring one kind of motion into another – for instance motion in a straight line into rotation or vice versa. This was an important problem for steam engines and sewing machines, among others. Mathematicians developed several kinds of linkages, for instance: Peaucellier-Lipkin linkage, Chebyshev's Lambda Mechanism, four-bar linkage, leg mechanisms.

2. *Tools and artifacts*. We use several types of tools, instruments, indicators, etc. in order to obtain data and to reason about data. Most commonly known examples of mathematical instruments are (the choice is a little bit *ad hoc*): compass, straight edge, ruler, pantograph, templates, protractor, French curve, lesbian curve, slide ruler, planimeter, trammel of Archimedes, tomahawk, perspectograph, integraph, spirograph, opisometer, calculators, abacus, etc. Some of these tools become obsolete recently and they are replaced by suitable computer software.
3. *Mathematical Mechanic*. Many interesting examples of intuitive explanations related to physics can be found in (Levi, 2009). Of course, argumentation based on a physical model can not replace a rigorous mathematical proof but it can still be very helpful in grasping mathematical ideas. Among the examples dicussed by Levi we found e.g. (Levi, 2009, p. 4):

   (a) The Pythagorean theorem can be explained by the law of conservation of energy.
   (b) Flipping a switch in a simple circuit proves the inequality $\sqrt{ab} \leqslant \frac{1}{2}(a+b)$.
   (c) Examining the motion of a bike wheel proves the Gauss-Bonnet formula (no prior exposure is assumed; all the background is provided).
   (d) Both the Cauchy integral formula and the Riemann mapping theorem (both explained in the appropriate section) become intuitively obvious by observing fluid motion.

The author claims that this physical approach: offers less computation, answers are given often conceptually, they can lead to new discoveries, they require not very sophisticated background.

The crucial thing in the examples of physical argumentations associated with a given mathematical result is a choice of an adequate model. For example, in results from of complex analysis (like, say, Cauchy Integral Formula or Riemann Mapping Theorem) it could be the fluid flow or heat flow. Center of mass may be useful in argumentations associated with theorems in geometry (remember Archimedes?), effects connected with gas pressure can be helpful in illustrations related to the curvature problems (like Gauss-Bonnet Theorem), etc.

4. *Mathematical constants calculated on an empirical basis.* Buffon's experiment about tossing a needle (and looking for the probability that it crosses parallel lines drawn on the paper) may make students reflect on how the famous mathematical constant $\pi$ is involved in an empirical process. (Galperin, 2003) analyzes the following experiment:

   > Let the mass of two balls be $M$ and $m$, respectively. Suppose that $M = 100^n m$ ($n \geqslant 0$). We roll the ball $M$ towards ball $m$ which is near the wall. Thus $M$ hits $m$ which bounces off the wall. How many collisions occur (jointly, i.e. between $M$ and $m$ and between $m$ and the wall) before the ball $M$ changes direction? The answer depends on $n$, of course.

   A surprising fact is that the number of balls' collisions is equal *precisely* to the first $n + 1$ digits of $\pi$. It is worth noticing that the result is purely *deterministic* and not based on probability. Another interesting feature of this problem lies in the fact that changing a perspective (in this case: from plain description of motions to the configuration space) essentially simplifies the solution of the problem.

5. *Supertasks.* By a supertask one means making an infinite number of steps in a finite amount of time. Supertasks have a long history, going back to Zeno's paradoxes of motion. They are still vividly discussed, also in new forms (e.g.: Thomson's lamp, Turing machines which can make an infinite number of steps in a finite amount of time, etc.). An interesting case of Laraudogoitia colliding balls (Laraudogoitia, 1996) is analyzed in (Romero, 2014) where its physical non-realizability is shown with a reference to the general relativity theory. We think that discussion of supertasks may be valuable in mathematical education, because it forces the students to reflect on the subtle connections between mathematical infinities and physical reality.

## 2.4. Internal explanations

More sophisticated mathematical ideas can be intuitively explained by reference to some simpler ideas already known to the students. Such explanations are possible because mathematics is a coherent unity. Despite existence of so many its subdisciplines results and concepts from one of them can be without collision applied to any other.

Internal explanations are typical in situations when we discuss more general concepts and point to analogies with simple cases. Operations on abstract entities (say, matrices or polynomials) are compared with operations on numbers. Other examples of such explanations are provided e.g. by illustrating arithmetical operations by suitably chosen geometric representations.

Let us look at three a little bit more sophisticated examples of intuitive internal explanations. Notice the style of argumentation in each case.

1. *Forcing.* The technique of forcing was developed by Paul Cohen (Cohen, 1966). It can be used for obtaining independence results in set theory. The technique is a little bit sophisticated but some textbooks (for instance (Koepke, 2013) which describes the example discussed below) associate intuitive explanations to its presentation. The concepts of forcing and generic model of set theory may be better understood when compared to the transcendental extensions of fields. First, let us recall that:

   (a) The field of complex numbers is algebraically closed. It contains the prime field of rational numbers $\mathbb{Q}$.
   (b) The field axioms do not decide the existence of, say, $\sqrt{2}$. It does not exist in $\mathbb{Q}$, but exists in $\mathbb{Q}(\sqrt{2})$. The number $\sqrt[3]{2}$ does not exist in field extensions of $\mathbb{Q}$ by square roots.
   (c) Given a field $k$ and an element $a$ which is transcendental over $k$ (i.e. such that $p(a) \neq 0$ for all $p \in k[x]$) one forms the extension $k(a)$ of $k$.
   (d) Taking into account the countable field $\mathbb{Q}$, one can construct a transcendental real number $a = 0, a_0 a_1 a_2 \ldots$ by successively choosing decimals $a_i$ in such a way that $0, a_0 a_1 a_2 \ldots a_m$ forces $p_n(a) \neq 0$, where $(p_n)$ is some enumeration of the polynomials from $k[x]$. In other words, this choice implies that:

   $$\forall b(b = 0, a_0 a_1 a_2 \ldots a_m b_{m+1} b_{m+1} \ldots \rightarrow p_n(b) \neq 0).$$

   In this formula we quantify over all $b = 0, b_0 b_1 b_2 \ldots b_m b_{m+1} b_{m+1} \ldots$ such that the first $m$ places of this expansion satisfy the equations: $b_i = a_i$ for all $1 \leqslant i \leqslant m$.
   (e) In the formulation of *forcing method* in set theory this last condition is expressed as:

   $$0, a_0 a_1 a_2 \ldots a_m \Vdash p_n(\dot{x}) \neq 0,$$

   where $\dot{x}$ is a *name* for the transcendental real to be constructed.

   Now, this situation is used to describe *intuitively* the method of construction of generic models in set theory:

   (a) We start with transitive submodels $(M, \in)$ of the standard universe $(V, \in)$.
   (b) We construct minimal submodels, thus similar to the prime field $\mathbb{Q}$.
   (c) We construct *generic extensions* $M \subseteq N$ by adjoining *generic* sets $G$, thus corresponding to transcendental numbers: $N = M[G]$.
   (d) $G$ is describable by infinitely many formulas in the ground model $M$. It is constructed in a countable recursive process with the use of countably many requirements which can be expressed in $M$.

   Thus, a new method created in set theory is explained by a reference to well known techniques from algebra. It should probably be added that the constructions in question involve also some extra logical or set theoretical results: the Downward Löwenheim-Skolem Theorem gives us countable models and the Mostowski Collapsing Lemma provides transitive models.

2. *Riemann hypothesis.* Davis and Hersh discuss cases where one can talk in some reasonable sense about *truth with probability one* (Davis, Hersh, 1981). Let us recall that the Möbius function $\mu(x)$ is defined as follows for any natural number $x$:

   (a) $\mu(x) = 0$, if $x$ is divisible by a square of some number;
   (b) $\mu(x) = 1$, if all the factors in the prime factorization of $x$ are different and there is an odd number of them;
   (c) $\mu(x) = -1$, if all the factors in the prime factorization of $x$ are different and there is an even number of them.

   Now, let Mertens function $M(x)$ be equal the sum of all $\mu(y)$, for $y \leqslant x$. It is known that the Riemann Hypothesis is equivalent to some condition describing the rate of growth of $M(x)$ as compared with $x^{\frac{1}{2}+\varepsilon}$, where $x \to \infty$ and $\varepsilon > 0$. There comes an *intuitive* assumption: let us think of $M(x)$ as a *random variable function*. Indeed, it seems that prime factorizations may contain even or odd number factors, without any regularity, and hence the function $\mu$ takes the values 1 and $-1$ with equal probability. Omitting the calculations that follow, let us come directly to the conclusion. It claims that the Riemann Hypothesis is true *with probability one*. For some readers this may sound reasonable (under the accepted probabilistic interpretation of the function $M$), but others may consider such a statement as pure absurd.
3. *Extremal axioms in set theory.* By extremal axioms one means axioms which were formulated in order to obtain a unique characterization of intended models. Examples are: the axiom of completeness in Hilbert's system of geometry, the axiom of induction in arithmetic, the axiom of continuity, Fraenkel's axiom of restriction in set theory, Gödel's axiom of constructibility, axioms of the existence of large cardinal numbers. Due to the famous *limitative theorems* in metalogic one recognizes the limitations of such axioms. Some of these axioms were rejected, partly for pragmatic reasons. For instance, Fraenkel's axiom of restriction, which by itself was a *minimality axiom* (claiming that there exist only such sets whose existence can be proved in set theory) was criticized and finally abandoned. The perspective in set theory has been changed in favor of axioms of *maximality*, i.e. axioms of the existence of large cardinal numbers. Besides some mathematical arguments supporting such a change, their proponents used also *intuitive arguments with a pragmatic flavor*: we want the universe of sets to be as rich as possible, similarly to the demand expressed in Hilbert's completeness axiom in geometry. Details concerning the rejection of restriction axioms in set theory can be found in (Fraenkel, Bar-Hillel, Levy, 1973).

## 3. Final remarks

The reflections presented above could be confronted with opinions expressed by mathematics educators. As already mentioned, we appreciate the approach of Anna Sierpińska (Sierpińska, 1994). Our proposals are also compatible, as it seems, with those expressed by David Tall (Tall, 2013). We appreciate especially his comments concerning *met-befores*, i.e. ideas and beliefs which a student brings with himself when he is entering the subsequent stages of mathematical education. Of some relevance to our approach are also other Tall's proposals – e.g. the intermediate role of *procepts* and the importance of *structural theorems* in mathematical education.

The ideas presented in this paper have been applied during classes devoted to mathematical problem solving whose participants were students of cognitive science. Our belief that intuitive explanations may be responsible for better understanding of mathematical ideas was supported by comments of the students and the content of their essays.

We have been working recently on a collection of mathematical problems which can presumably be useful for the shaping of correct mathematical intuitions. Besides the proposals of Sierpińska and Tall mentioned above we take into account also the classical ideas of Polya concerning the strategies of problem solving (Polya, 2009, 2014) and Schoenfeld concerning metacognitive control in problem solving (Schoenfeld, 1985). The collection in question is composed mainly of problems which show the often illusory character of immediate solutions accepted without reflection, on the basis of a common day experience.

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