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## From Euclid's *Elements* to the methodology of mathematics. Two ways of viewing mathematical theory\*

**Abstract.** We present two sets of lessons on the history of mathematics designed for prospective teachers: (1) Euclid's Theory of Area, and (2) Euclid's Theory of Similar Figures. They aim to encourage students to think of mathematics by way of analysis of historical texts. Their historical content includes Euclid's *Elements*, Books I, II, and VI. The mathematical meaning of the discussed propositions is simple enough that we can focus on specific methodological questions, such as (a) what makes a set of propositions a theory, (b) what are the specific objectives of the discussed theories, (c) what are their common features.

In spite of many years' experience in teaching Euclid's geometry combined with methodological investigations, we cannot offer any empirical findings on how these lectures have affected the students' views on what a mathematical theory is. Therefore, we can only speculate on the hypothetical impact of these lectures on students.

### 1. Introduction

Scholars in the foundations of mathematics share the view that a theory is a set of sentences that follow from a group of axioms. Ironically, they present Euclid's *Elements* as a model historical example of such a system (see Hilbert, 1922; Barwise, 1999). This methodology focuses on axioms which are considered to be the very first mathematical truths, while propositions are viewed in terms of entailment alone. Consequently, it does not provide any means to decide whether one proposition is more important than another, or what a theory as a whole is for. Both school and academic textbooks mimic that attitude, while at the same time not providing a full set of axioms, e.g. textbooks do not usually consider all axioms

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\*2010 Mathematics Subject Classification: Primary: 97G40; Secondary: 03A05

Keywords and phrases: Elements, theory, theorem, triangulation, similar figures, Pythagorean theorem.

for real numbers, except for course books in mathematical logic or model theory. As a result, students implicitly adopt a view that mathematics is but an endless chain of theorems governed by logical consequence with no beginning (axioms) and no actual aims.

Contrarily, in mathematical practice, sentences which are grouped in a theory such as calculus, algebra, or more specific ones like Fourier series, form a hierarchical structure; sometimes the unique role of a theorem is even mirrored by its name, e.g. the fundamental theorem of calculus, or the fundamental theorem of algebra. Hence, in our view, a theory is a group of axioms, definitions, and theorems designed to solve a problem; in addition, these theorems share the same methodology characterized in terms of mathematical tricks. Throughout our lessons, we develop a methodology that seeks to adhere to mathematical practice rather than the axiomatic philosophy.

In teachers' training, elementary geometry is a way of introducing the methodology of mathematics to students, specifically theorem proving. While the mathematics used in the course is simple, a lecturer, i.e. a teacher of prospective teachers, can focus on the relationship between the premises and conclusions; he can point out references to the axioms and previously proved theorems; he can also highlight the use of undefined terms, and explain the difference between direct and indirect proof. By and large, Euclidean geometry is presented from the perspective of the 20<sup>th</sup> century philosophy of mathematics as a set of sentences that follow from axioms.

We offer an alternative view on mathematical theory, namely: Theory is a hierarchical structure of theorems designed to solve a specific problem and characterized by having the same methodology. We accept the general idea of the deductive nature of mathematics, however, in this context, by methodology we mean a set of so-called mathematical tricks irreducible to logical consequence. To illustrate this new perspective, in sections 3 and 4, we present two theories identified in Euclid's *Elements*: Theory of Area and Theory of Similar Figures. We present them as hierarchical systems crowned by proposition II.14 (i.e. *Elements*, Book II, proposition 14) in the case of the former theory, and VI.31 in the case of the later. We also reveal a technique of triangulation, and show that it is a common feature of these theories. Indeed, the triangulation enables to reduce problems concerning polygons to triangles. Nevertheless, this technique is covered neither by modern axiomatic analyses of elementary geometry nor by the mathematics curriculum. In the section 5, we offer diagrams representing the triangulation method as it relates to the Pythagorean theorem. We show that depending on whether the triangulation is applied within the Theory of Area or the Theory of Similar Figures, it gives different proofs of the Pythagorean theorem, namely I.47 and VI.31 respectively. In section 6, we discuss some routine methodological and metamathematical issues such as relationship between concepts of equal areas, the role of Archimedean axiom in the deductive structure of discussed theories, and alleged generalizations of the Pythagorean theorem. In this way, we show how starting with elementary geometry one can introduce topics of modern methodology of mathematics.

## 2. Geometry without numbers

The ancient Greeks developed geometry without real numbers, resulting in lack of number representation for length of line segment, area of figure, volume of solid, or measure of angle<sup>1</sup>. Nevertheless, the Greeks created their own techniques to deal with geometric objects themselves, mainly the Theory of Area and the Theory of Similar Figures. Therefore, to study Euclid's *Elements* in terms of ancient Greek mathematics, one has to put away the modern techniques relying on the facts that line segments have lengths represented by real numbers, triangles have areas calculated by the formula  $\frac{1}{2}ah$  or  $\frac{1}{2}absin\alpha$ , etc. Generally, within the Theory of Area, a triangle is related to another figure, e.g. to a square, rather than to a number; within the theory of Similar Figures, e.g. similar triangles,  $T_1, T_2$  are related to one another by proportion  $T_1 : T_2 :: a : b$ , where  $a, b$  are line segments.

In what follows, we present a simplified, non-historian-friendly version of Euclid's geometry that swings in-between ancient notions and its modern counterparts. Note, however, that usually while replacing a triangle by its area is almost unnoticeable, the interpretation of a proportion like  $a : b :: c : d$  as an equality of fractions (of real numbers)  $\frac{a}{b} = \frac{c}{d}$  makes a crucial difference. Although Euclid's ratio is defined in book V, it acquires a mathematical meaning only as a component of proportion. Unlike the fractions, ratios are not subject to operations of addition and multiplication. To give an example, in the arithmetic of fractions,  $\frac{a}{b}$  multiplied by  $\frac{b}{c}$  gives  $\frac{a}{c}$ . In the *Elements*, proposition V.22 boils to the effect that if  $\frac{a}{b} = \frac{d}{e}$  and  $\frac{b}{c} = \frac{e}{f}$ , then  $\frac{a}{c} = \frac{d}{f}$ . Now, even if ratio  $\frac{a}{c}$  is interpreted as the result of multiplication, it still has to be a part of the proportion  $a : c :: d : f$ . Consequently, in the Greeks mathematics, it was not an easy task to formulate a relation between similar figures, while in modern mathematics, it is a simple statement: (areas of) similar figures are to one another as the square of the similarity scale. In Greek mathematics, "the similarity scale" was represented by the proportion of corresponding sides of similar figures, nevertheless the square of "the similarity scale" had to be represented by a very intricate proportion (see *Elements*, VI.19 or section 4 below). Therefore, our simplifications concern notations and symbolic representations, rather than techniques of Greek mathematics.

Although we represent some proportions by equalities of fractions, we are aware of the constrains of the Greek theory, specifically that proportions can not be transformed as easily as our modern fractions. Moreover, while arithmetic of fractions follows from the axioms of ordered field, "arithmetic of ratios" is developed in the *Elements* book V. As a result, e.g. the simple fact, if  $\frac{a}{c} = \frac{b}{c}$ , then  $a = b$ , obtains in any field, whether it is Archimedean or non-Archimedean field, while its ancient counterpart covered by the proposition V.9, if  $a : c :: b : c$ , then  $a = b$ , depends on the Archimedean axiom.

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<sup>1</sup>We recommend O'Leary,(2010), ch. 5) as a general overview of Euclid's *Elements*, still it covers neither the technique of triangulation nor the proportion theory.

### 3. Euclid’s Theory of Area

Euclid’s Theory of Area is a set of propositions and constructions which allow to transform a polygon (in Euclid’s words: *area, figure, or form*)  $A$ , into a square  $S$ . The equality  $A = S$  is founded on the *Common Notions* axioms (abbreviated as CN) and straightedge and compass constructions as characterized in *Postulates* 1 to 3. Here are the relevant Common Notions: 1. *Things equal to the same thing are also equal to one another.* 2. *And if equal things are added to equal things then the wholes are equal.* 3. *And if equal things are subtracted from equal things then the remainders are equal.* 4. *And things coinciding with one another are equal to one another.*<sup>2</sup>

The theory of area starts with proposition I.35 stating the equality of parallelograms  $ADCB$  and  $EFCE$  which are on the same base and between the same parallels, i.e. with the same height (see Figure 1, upper left diagram). In propositions I.1 through I.34, the equality of figures means congruence. Still, in I.35, Euclid implicitly adopts another meaning of equality, namely, he introduces the equality of non-congruent figures. The proof of I.35 proceeds as follows: Triangles  $AEB$ ,  $DFC$  are congruent,  $AEB \cong DFC$ . When triangle  $DEG$  is subtracted from each of them, the remainders  $ADGB$  and  $EFCE$  are equal by CN3; equal in a new sense, which we represent by the formula  $ADGB = EFCE$ . When triangle  $GCB$  is added to  $ADGB$ ,  $EFCE$ , the whole parallelogram  $ABCD$  is equal to the whole parallelogram  $EFCE$  by CN2.

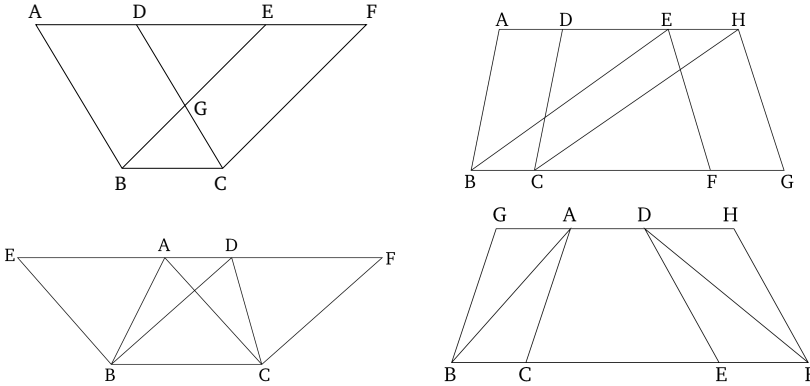


Fig. 1. *Elements*, I.35–38

In proposition I.36 (Figure 1, upper right diagram), Euclid shows that parallelograms on congruent bases are equal, i.e.  $ADCB = EHGF$ . His argument relies on the transitivity of equality guaranteed by CN1 and the parallelogram  $EHCB$ .

Proposition I.37 (Figure 1, lower left diagram) states that triangles on the same base and between the same parallels are equal,  $ACB = DCB$ . The proof of the proposition is this: since figures  $EACB$ ,  $DFCB$  are equal, their halves, i.e. triangles  $ACB$ ,  $DCB$  are equal too. Proposition I.38 (Figure 1, lower right diagram) reiterates that claim in regard to triangles on congruent bases,  $ACB = DFE$ . Euclid’s proofs of both propositions refer to I.34 which reads as follows:

<sup>2</sup>All translations of the *Elements* are from (Fitzpatrick, 2007).

a triangle is half of a parallelogram, e.g. in the case of I.38, the triangle  $ACB$  is half of the parallelogram  $GACB$ .

Proposition I.42 provides a construction of a parallelogram equal to the given triangle  $ACB$ . Its proof consists of finding the midpoint  $E$  on the base  $BC$ . In the parallelogram  $FGCE$ , the angle  $FEC$  is to be congruent to the given angle  $D$ . To put it simply, let us assume  $D$  is the right angle. Thus, by proposition I.42, a triangle can be transformed into a rectangle. Note, however, that its height equals the height of the triangle, since both the triangle  $ACB$  that is to be transformed and the resulting rectangle  $FGCE$  are between the same parallels. In the next proposition, Euclid tackles this problem and provides a construction that transforms a given parallelogram into an equal parallelogram but with one side fixed at will. As a result, a triangle can be transformed into an equal rectangle, while the heights of these figures differ.

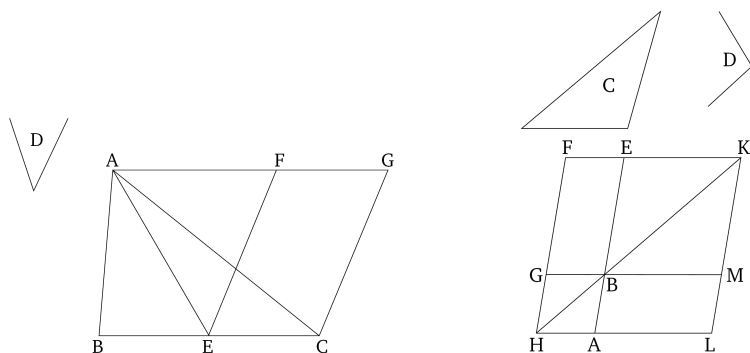


Fig. 2. *Elements*, I.42 (left), and I.44 (right)

On the diagram that accompanies proposition I.44 (see Figure 2 above), the triangle  $C$  equals the rectangle  $FEBG$ . Now, by CN2, rectangles  $FEBG$  and  $BMLA$  are equal, and by CN1, the triangle  $C$  is equal to the rectangle  $BMLA$ . This construction is known as “applying”, for it shows how to *apply a parallelogram equal to the given triangle  $C$  to the given straight-line  $AB$* .

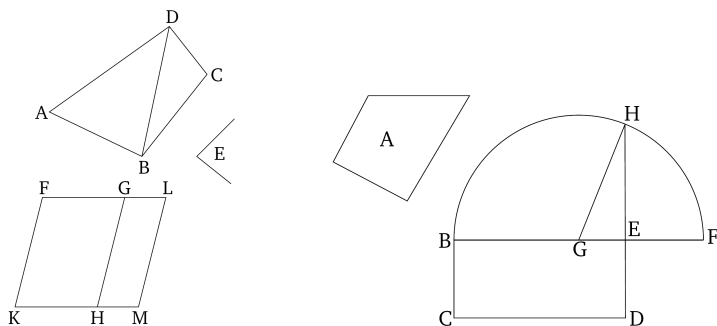


Fig. 3. *Elements*, I.45 (left), and II.14 (right)

Proposition I.45 summarizes a method we call the triangulation of polygons. Euclid's diagram presents a quadrangle  $ADCB$ , nevertheless, the method applies to

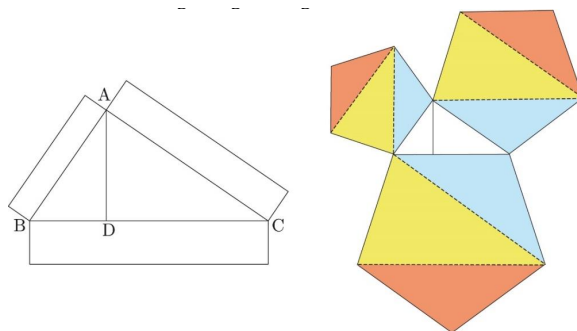
any polygon. The idea is this: divide the polygon  $ADCB$  into adjacent triangles, say  $ADB$ ,  $DCB$ ; by proposition I.41, transform each triangle into a parallelogram, say  $P_1$ ,  $P_2$ ; let us assume  $P_1$  is simply  $FGHK$ ; apply to the line  $GH$  a parallelogram  $GLMH$  equal to  $P_2$ . It easily follows that  $FLMK = ADCB$ . Then, the resulting parallelogram  $FLMK$  is transformed into a rectangle. In this way, any polygon  $A$  can be transformed into an equal rectangle. The theory of area culminates in a construction of squaring a figure introduced by proposition II.14 (see Figure 3 above).

By the triangulation and construction in I.45, the polygon  $A$  is transformed into the rectangle  $BEDC$ , then the construction II.14 proceeds like this: produce the line segment  $BE$ , make  $EF \equiv ED$ ; find  $G$ , the midpoint of the segment  $BF$ ; draw the semicircle  $BHF$  with the center  $G$  and the radius  $GB$ ; draw  $EH$  perpendicular to  $BF$ . It is shown that the square on  $EH$  (in short,  $\text{sq}EH$ ), equals the rectangle  $BEDC$ .

To sum up, Euclid's Theory of Area consists of constructions that transform a polygon  $A$  into a square  $S$  such that the equality  $A = S$  holds. These constructions are included in the propositions that guarantee the equality of the relevant figures.

#### 4. Euclid's theory of similar figures

Euclid's theory of similar figures builds on his theory of proportion as developed in Book V. It culminates in proposition VI.31, stating that *In right-angled triangles, the figure on the side subtending the right-angle is equal to the similar, and similarly described, figures on the sides surrounding the right-angle.* In what follows, we focus on a special case of VI.31, with similar pentagons drawn on the sides of a right-angle triangle to reveal the method of triangulation.



**Fig. 4.** *Elements*, VI.31 (left). Similar pentagons on sides of a right-angle triangle (right)

We start with proposition VI.14 (see Figure 5 below). It could be viewed as the ancient counterpart of the modern formulae  $ab \sin \alpha$  for the area of a parallelogram. This proposition states, that if the proportion  $FB : BG :: EB : BD$  obtains, then the parallelograms  $ECGB$  and  $FBDA$  are equal. Similarly, in the next proposition, Euclid shows that if proportion  $BA : AE :: DA : AC$  obtains, then triangles  $BCA$  and  $AED$  are equal.

Propositions VI.14 and VI.15 are applied in the proof of VI.19 (see Figure 6 below), which encodes a truth known in modern mathematics as a relationship between areas of similar triangles and their similarity scale. While the notion of the similarity scale cannot be expressed in Euclid's theory of proportion, the thesis of proposition VI.19 seems to be a bit murky, namely, it states that *Similar triangles are to one another in the duplicate ratio of corresponding sides*.

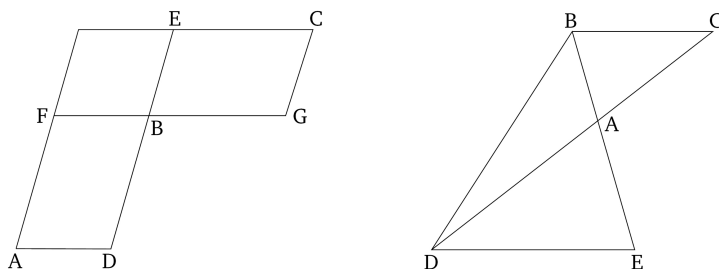


Fig. 5. *Elements*, VI.14 (left), VI.15 (right)

Since triangles  $ACB$  and  $DFE$  are similar, proportions  $AB : DE :: BC : EF$  and  $BC : EF :: EF : BG$  obtain. As to the latter, point  $G$  is constructed in such a way that the proportion  $BC : EF :: EF : BG$  obtains. To put it another way, the line  $BG$  is the so-called third proportional between  $BC$  and  $EF$ ; proposition VI.12 introduces the construction of the *third proportional*. From the above proportions, it follows that  $AB : DE :: EF : BG$ , and by VI.15, the triangle  $AGB$  equals the triangle  $DFE$ . By VI.2, the triangle  $ACB$  is to the triangle  $AGB$  as the line  $BC$  is to  $BG$ . Since  $AGB$  equals  $DFE$ , the proportion  $\triangle ACB : \triangle DFE :: BC : BG$  obtains.

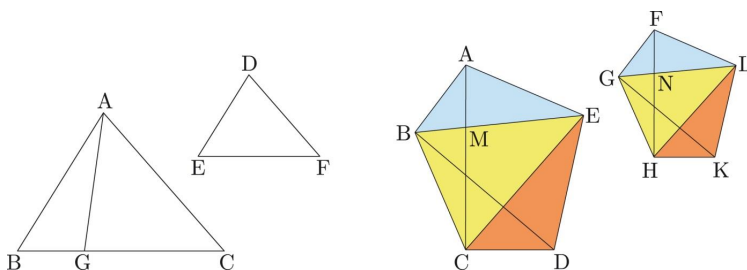


Fig. 6. *Elements*, VI.19 (left), VI.20 (right)

In modern exposition of the proportion theory, the similarity scale of triangles  $ACB$  and  $DFE$  is represented by the fraction  $\frac{a}{b}$ , where  $a$  stands for the measure of the line  $AB$ , and  $b$  for the measure of the line  $ED$ . Supposing that  $AB : DE = BC : EF = \frac{a}{b}$ , we obtain the following “proportions”:

$$BC : BG = (BC : EF)(EF : BG) = \frac{a}{b} \frac{a}{b}.$$

Thus,  $\triangle ACB : \triangle DFE = \frac{a}{b} \frac{a}{b}$ . Arguably, it represents the statement: *Similar triangles are to each other as the square of the similarity scale*.

Furthermore, since the proportion between similar triangles is established, Euclid can generalize this result to polygons through the triangulation. The technique of triangulation is applied in proposition VI.20, which reads: *Similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a duplicate ratio with respect to a corresponding side.*

The first part of this proposition is represented by the colors on the above diagram. In what follows, we apply a simplified version of VI.19. The similarity scale of pentagons can be represented by equal fractions  $AB : GF$ , or  $BE : GL$ , or  $CE : HL$ . By VI.19 we have,

$$\triangle AEB : \triangle FLG = (BE : GL)(BE : GL),$$

$$\triangle BEC : \triangle GLH = (BE : GL)(BE : GL),$$

$$\triangle EDC : \triangle LKH = (CE : HL)(CE : HL).$$

Since  $CE : HL = BE : GL$ , by V.12 it follows from these proportions that

$$(\triangle AEB + \triangle BEC + \triangle EDC) : (\triangle FLG + \triangle GLH + \triangle LKH) = (BE : GL)(BE : GL).$$

Finally,

$$AEDCB : FLKHG = (AB : GF)(AB : GF).$$

Going back to the proposition VI.31 and the pentagons represented in Figure 4, we can show that the blue triangle described on the side  $AB$  is to the blue triangle on the side  $BC$  as  $BD$  is to  $BC$ . In the same manner, the blue triangle on the side  $AC$  is to the blue triangle on the side  $DC$  as  $DC$  is to  $BC$ . The same applies to the yellow and red triangles. Thus, in the very special case of similar figures on the sides of a right-angle triangle, the “square of similarity scale” is represented by the ratio  $BD : BC$  and  $DC : BC$  respectively. Euclid demonstrates this relation in proposition VI.9. Now, “adding up” the above proportions, we get

$$(\text{pentagon on } AB + \text{pentagon on } AC) : (\text{pentagon on } BC) = (BD + DC) : BC.$$

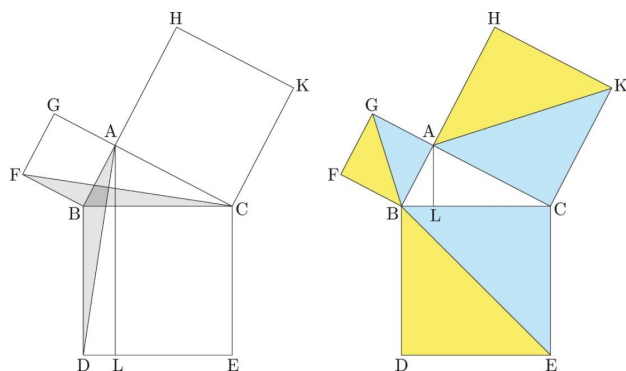
Since  $BD + DC = BC$ , we finally obtain

$$(\text{pentagon on } AB + \text{pentagon on } AC) = (\text{pentagon on } BC).$$

## 5. Educational recourse to diagrams

Euclid’s *Elements* provide two proofs of the Pythagorean Theorem, namely I.47 and VI.31. Both of them apply the triangulation method that is represented in Figure 7. In I.47, the square  $FGAB$  equals the rectangle  $BDL$  due to congruent grey triangles. Thus, within the Theory of Area, triangles transfer the equality of figures, namely:  $sq(FGAB) = 2\triangle FCB = 2\triangle ABD = \text{rectangle}(BLD)$ . The second part of the proof, that is the equality  $sq(AHKC) = \text{rectangle}(CEL)$ , proceeds in the same manner.





**Fig. 7.** *Elements*, I.47 (left), VI.31 (right)

In VI.31, squares on  $BA$  and  $BC$  are similar, and the “square” of similarity scale is represented by the ratio  $BL : BC$ . In the same manner, squares on  $AC$  and  $BC$  are similar, and the “square” of similarity scale is represented by the ratio  $LC : BC$ . The blue and yellow triangles underline the fact that these relations have been established due to triangulation, namely  $\triangle GAB : \triangle BCE : BL : BC$  and  $\triangle FGB : \triangle DBE : BL : BC$ , therefore,

$$sq(FGAB) : sq(DBCE) :: BL : BC.$$

Similarly,  $\triangle AKC : \triangle BCE : LC : BC$  and  $\triangle AHK : \triangle BDE : LC : BC$ . Therefore,

$$sq(AHKC) : sq(BCED) :: LC : BC.$$

“Adding up” these proportions, by V.24, we obtain the following proportion

$$sq(FGAB) + sq(AHKC) : sq(BCED) :: BC : BC.$$

Finally, by V.16 and V.9, the equality  $sq(FGAB) + sq(AHKC) = sq(BCED)$  holds.

By comparing these proofs, we can reveal yet another phenomena, this time of cognitive, rather than strictly mathematical nature. The proof of proposition I.47 is based on a partition of the square  $BCED$  into rectangles  $BLD$  and  $CEL$ ; it demonstrates that  $sq(FGAB) = \text{rectangle}(BLD)$ , and  $sq(AHKC) = \text{rectangle}(LEC)$ ; these equalities are easily represented on a diagram. On the other hand, in the proof of proposition VI.31, squares  $FGAB$  and  $AHKC$  can not be represented by any parts of the square  $BCED$ . Nevertheless, we can represent the relationship between these squares by the formula

$$sq(FGAB) + sq(AHKC) = sq(BCED).$$

The sign  $+$  finds no diagrammatic counterpart; in fact, there is no reference to the addition in the enunciation of the proposition VI.31; it reads: *figure on BC is equal to similar, and similarly described figures on BA, AC*. Yet, the sign  $+$  is understandable on the theoretical level, within the proportion theory, specifically through the proposition V.24. It reads: *the first and the fifth, added together, AG,*

will also have the same ratio to the second  $C$  that the third and the sixth,  $DH$ , has to the fourth  $F$ . That is, if  $AB : C :: DE : F$ , and  $BG : C :: EHF$ , then  $AB : C :: DH : F$ . Since throughout the whole book V magnitudes are represented by line segments, the equality  $AG = AB + BG$  to which the word *added* applies, is represented by line segments. However, the addition of squares is not represented by any portion of the square  $BCED$ ; it can be represented but by the formula

$$sq(FGAB) + sq(AHKC).$$

## 6. Comparing definitions and proofs

Presented above sets of lessons offer the opportunity to introduce metamathematical techniques of comparing definitions and axiomatic backgrounds of the proofs. Here is a sketch of what can be studied, we skip here the question how these topics could be studied.

Approximately 400 proofs of the Pythagorean Theorem circulate on the Internet (see Maor, 2007). They can all be divided into three groups due to the applied method as follows: 1) cut-and-paste proofs, 2) proofs based on *Elements*, I.47, 3) proofs based on *Elements*, VI.31. Most of them are of the cut-and-paste kind, since they build on the dissection of squares into congruent triangles. Thus, they implicitly apply a specific notion of the equality of figures. Since courses in elementary geometry do not include any reference to the complete theory of equality founded on dissection, like the one developed in (Hilbert, 1970, ch. 4), the only way students can follow along while considering these proofs is to meditate on diagrams. Unintentionally, they imitate the 12<sup>th</sup> century Hindu dissection-proof in which the inferential knowledge is reduced to just one word: “See” (see Maor, 2007, p. 65). It can be shown, however, that equality founded on dissection and Euclid’s equality of figures are two different concepts, i.e. there is a model of an Euclidean plane with triangles equal in Euclid’s sense and not equal in terms of dissection (see Hilbert, 1970). This claim seems to contradict the Bolyaia-Gerwien theorem that boils down to the fact that equality based on dissection is equivalent to Euclid’s equality of figures (in standard wording, to the  $ah/2$  formula for the area of triangle). Indeed, the Bolyaia-Gerwien theorem does not hold on non-Archimedean planes, while Euclid’s Theory of Area is valid on (some) non-Archimedean planes, consequently the proof of the proposition I.47 can be reconstructed on a non-Archimedean plane<sup>3</sup>.

The proof of VI.31 builds on the proportion theory, as a result, on the Archimedean axiom (definition V.4), therefore it can be completed only on (some) Archimedean planes. In fact, definition V.4 is referred to only once in book V,

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<sup>3</sup>Cf. J. Baldwin’s claim: “Euclid’s proof of Pythagoras’s theorem I.47 uses the properties of area [...]. His second proof [...] uses the property of similar triangles [...]. In both cases Euclid depends on the theory of proportionality (and thus implicitly on Archimedes’ axiom) to prove the Pythagorean theorem” (Baldwin, 2018. p. 368). In section 3, we have shown that Euclid’s Theory of Area does not refer to proportions.

namely in the proposition V.8. Still, through the appropriate models we can also show that Euclid's use of definition V.4 in the proof of V.8 is essential (see Błaszczyk, Mrówka, 2013, pp. 176–184)<sup>4</sup>.

Proposition VI.31 is often called a generalization of I.47. George Polya, for example, seeks to support this claim with a kind of heuristic argument. In fact, his approach implicitly applies proposition VI.19 and in this way it depends on the Archimedean axiom (see Polya, 1957, pp. 12–17). Although the theses of the propositions VI.31 and I.47 could be viewed in terms of generality, the proof of VI.31 is by no means more general than the proof of I.47.

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<sup>4</sup>All throughout his Baldwin, 2018 John Baldwin reiterates the claim that Euclid's theory of proportion "depends on Archimedes' axiom". However, he just states that book V includes definition 4 rather than provide a proof to the metamathematical result that Euclid's theory depends on definition V.4.